

NERS 312

Elements of Nuclear Engineering and Radiological Sciences II

aka **Nuclear Physics for Nuclear Engineers**

Lecture Notes for Chapter 15: β decay

Supplement to (Krane II: Chapter 9)

The lecture number corresponds directly to the chapter number in the online book.

The section numbers, and equation numbers correspond directly to those in the online book.

A short illustrated story on the life of a neutron



from P. J. Fournier's "What the Quark" website:
<http://www.whatthequark.com/>
devoted to "Cartoons about Life, Science, and What Not".

Chapter 15: In this Chapter you will learn

Chapter 15.0: Basic introduction to β decay

- The three views of β decay
- Consequences of β -decay's 3-body final state

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- Kurie plots: Shape of the β spectrum
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Chapter 15: β decay

β -particle's are either electrons¹ or positrons that are emitted through a certain class of nuclear decay associated with the “weak interaction”.

The discoverer of electrons was Henri Becquerel, who noticed that photographic plates, covered in black paper, stored near radioactive sources, became fogged.

The black paper (meant to keep the plates unexposed) was thick enough to stop α -particles, and Becquerel concluded that fogging was caused by a new form of radiation, one more penetrating than α -particles

The name “ β ”, followed naturally as the next letter in the Greek alphabet after α , α -particles having already been discovered and named by Rutherford.

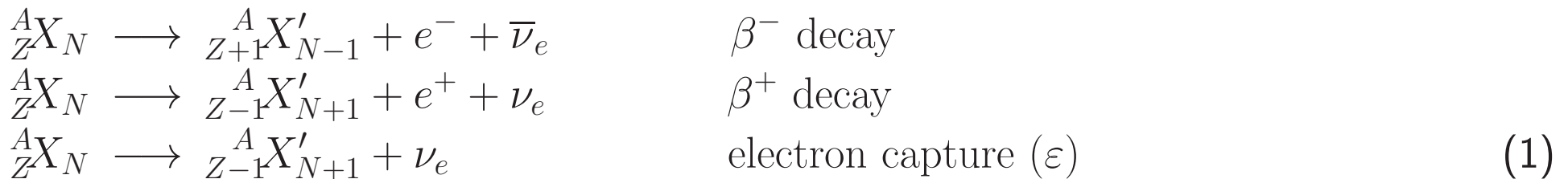
¹Technically, the word “electron” can represent either a negatron (a fancy word for e^-) or a positron (e^+). I'll use “electron” interchangeably with this meaning, and also e^- . Usually the context determines the meaning.

Since that discovery, we have learned that β -particles are about 100 times more penetrating than α -particles, and are spin- $\frac{1}{2}$ fermions.

Associated with the electrons is a conserved quantity, expressed as the quantum number known as the *lepton number*.

The lepton number of the negatron is, by convention +1. The lepton number of the positron, also the anti-particle² of the negatron, is -1.

Thus, in a negatron-positron annihilation event, the net lepton number is zero. Only leptons can carry lepton number. (More on this soon.) Recall, from Chapter 13 (Chapter 6 in Krane), our discussion of the various decay modes that are associated with β decay:



²An anti-particle has the reverse signs of all the quantum numbers of its particle counterpart. When particle-particle annihilation occurs, all that remains is energy, momentum, and angular momentum, as the sum of all quantum numbers must be zero.

We see from these processes that there are other particles called *neutrinos*.

Neutrinos are also spin- $\frac{1}{2}$ leptons (part of the larger fermion family). They are very nearly massless (but proven to have mass³).

The electron neutrino is given the symbol ν_e , and has lepton number $+1$. The antineutrino, the $\bar{\nu}_e$, has lepton number -1 . A sketch of the organization of fundamental particles is given in Figure 1.

³A direct measurement of neutrino mass suggests that its upper limit is $m_{\nu_e} < 2.2\text{eV}$. Indirect measurement of the neutrino mass suggest that $0.04\text{eV} < m_{\nu_e} < 0.3\text{eV}$. For the more massive lepton family groups, $m_{\nu_\mu} < 180\text{keV}$, and $m_{\nu_\tau} < 15.5\text{MeV}$.

Standard Model of FUNDAMENTAL PARTICLES AND INTERACTIONS

The Standard Model summarizes the current knowledge in Particle Physics. It is the quantum theory that includes the theory of strong interactions (quantum chromodynamics or QCD) and the unified theory of weak and electromagnetic interactions (electroweak). Gravity is included on this chart because it is one of the fundamental interactions even though not part of the "Standard Model."

FERMIONS

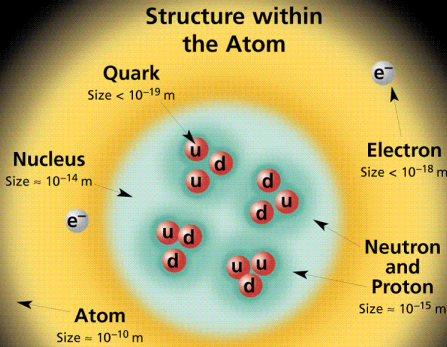
matter constituents
spin = 1/2, 3/2, 5/2, ...

Leptons spin = 1/2			Quarks spin = 1/2		
Flavor	Mass GeV/c ²	Electric charge	Flavor	Approx. Mass GeV/c ²	Electric charge
ν_e electron neutrino	$<1 \times 10^{-8}$	0	u up	0.003	2/3
e electron	0.000511	-1	d down	0.006	-1/3
ν_μ muon neutrino	<0.0002	0	c charm	1.3	2/3
μ muon	0.106	-1	s strange	0.1	-1/3
ν_τ tau neutrino	<0.02	0	t top	175	2/3
τ tau	1.7771	-1	b bottom	4.3	-1/3

Spin is the intrinsic angular momentum of particles. Spin is given in units of \hbar , which is the quantum unit of angular momentum, where $\hbar = h/2\pi = 6.58 \times 10^{-25}$ GeV s = 1.05×10^{-34} J s.

Electric charges are given in units of the proton's charge. In SI units the electric charge of the proton is 1.60×10^{-19} coulombs.

The **energy** unit of particle physics is the electronvolt (eV), the energy gained by one electron in crossing a potential difference of one volt. **Masses** are given in GeV/c² (remember $E = mc^2$), where 1 GeV = 10^9 eV = 1.60×10^{-10} joule. The mass of the proton is 0.938 GeV/c² = 1.67×10^{-27} kg.



If the protons and neutrons in this picture were 10 cm across, then the quarks and electrons would be less than 0.1 mm in size and the entire atom would be about 10 km across.

BOSONS

force carriers
spin = 0, 1, 2, ...

Unified Electroweak spin = 1			Strong (color) spin = 1		
Name	Mass GeV/c ²	Electric charge	Name	Mass GeV/c ²	Electric charge
γ photon	0	0	g gluon	0	0
W^-	80.4	-1			
W^+	80.4	+1			
Z^0	91.187	0			

Color Charge

Each quark carries one of three types of "strong charge," also called "color charge." These charges have nothing to do with the colors of visible light. There are eight possible types of color charge for gluons. Just as electrically-charged particles interact by exchanging photons, in strong interactions color-charged particles interact by exchanging gluons. Leptons, photons, and W and Z bosons have no strong interactions and hence no color charge.

Quarks Confined in Mesons and Baryons

One cannot isolate quarks and gluons; they are confined in color-neutral particles called **hadrons**. This confinement (binding) results from multiple exchanges of gluons among the color-charged constituents. As color-charged particles (quarks and gluons) move apart, the energy in the color-force field between them increases. This energy eventually is converted into additional quark-antiquark pairs (see figure below). The quarks and antiquarks then combine into hadrons; these are the particles seen to emerge. Two types of hadrons have been observed in nature: **mesons** $q\bar{q}$ and **baryons** qqq .

Residual Strong Interaction

The strong binding of color-neutral protons and neutrons to form nuclei is due to residual strong interactions between their color-charged constituents. It is similar to the residual electrical interaction that binds electrically neutral atoms to form molecules. It can also be viewed as the exchange of mesons between the hadrons.

PROPERTIES OF THE INTERACTIONS

Baryons qqq and Antibaryons $\bar{q}\bar{q}\bar{q}$					
Baryons are fermionic hadrons. There are about 120 types of baryons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c ²	Spin
p	proton	uud	1	0.938	1/2
\bar{p}	anti-proton	$\bar{u}\bar{u}\bar{d}$	-1	0.938	1/2
n	neutron	udd	0	0.940	1/2
Λ	lambda	uds	0	1.116	1/2
Ω^-	omega	sss	-1	1.672	3/2

Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic	Strong	
	Mass - Energy	Flavor	Electric Charge	Fundamental	Residual
Acts on:	Mass - Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles mediating:	Graviton (not yet observed)	W^+ W^- Z^0	γ	Gluons	Mesons
Strength relative to electromag for two u quarks at:		0.8	1	25	Not applicable to quarks
	10^{-18} m	10^{-4}	1	60	
	3×10^{-17} m	10^{-7}	1	Not applicable to hadrons	
for two protons in nucleus	10^{-41} 10^{-41} 10^{-36}				20

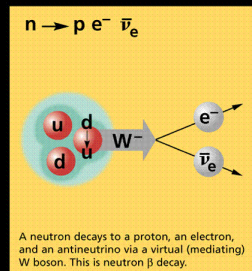
Mesons $q\bar{q}$					
Mesons are bosonic hadrons. There are about 140 types of mesons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c ²	Spin
π^+	pion	$u\bar{d}$	+1	0.140	0
K^-	kaon	$s\bar{u}$	-1	0.494	0
ρ^+	rho	$u\bar{d}$	+1	0.770	1
B^0	B-zero	$d\bar{b}$	0	5.279	0
η_c	eta-c	$c\bar{c}$	0	2.980	0

Matter and Antimatter

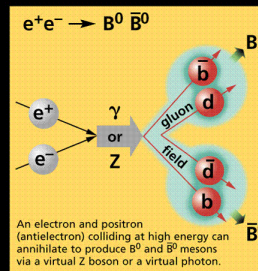
For every particle type there is a corresponding antiparticle type, denoted by a bar over the particle symbol (unless + or - charge is shown). Particle and antiparticle have identical mass and spin but opposite charges. Some electrically neutral bosons (e.g., Z^0 , γ , and $\eta_c = c\bar{c}$, but not $K^0 = d\bar{s}$) are their own antiparticles.

Figures

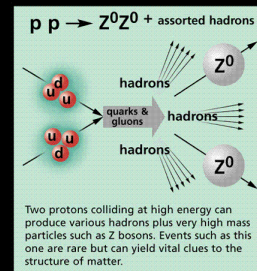
These diagrams are an artist's conception of physical processes. They are not exact and have no meaningful scale. Green shaded areas represent the cloud of gluons or the gluon field, and red lines trace the quark paths.



A neutron decays to a proton, an electron, and an antineutrino via a virtual (mediating) W^- boson. This is neutron β decay.



An electron and positron (antielectron) colliding at high energy can annihilate to produce B^0 and \bar{B}^0 mesons via a virtual Z boson or a virtual photon.



Two protons colliding at high energy can produce various hadrons plus very high mass particles such as Z bosons. Events such as this one are rare but can yield vital clues to the structure of matter.

The Particle Adventure

Visit the award-winning web feature *The Particle Adventure* at <http://ParticleAdventure.org>

This chart has been made possible by the generous support of:

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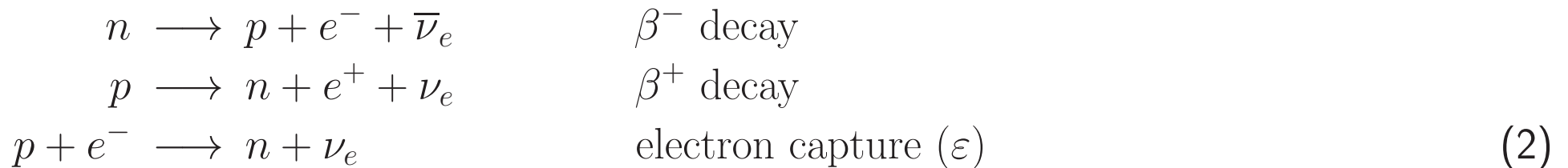
<http://CPEPweb.org>

Three views of β decay

There are three ways of viewing β decay.

The first is the “radiological physics view” expressed by (1).

The next is the “nuclear physics view”, where we recognize that the decays of the nuclei are actually caused by transformations of the nucleon constituents, as expressed in (2).



A free neutron will decay with a meanlife, $\tau = 885.7(8)\text{s}$, about 11 minutes.

A free proton is basically stable. Once these nucleons are bound in a nucleus, however, conservation of energy, with the availability of lower energy states, dictates whether or not these processes are free to proceed.

Then, there is the more microscopic view, the “particle physics view” expressed in (3),

$$\begin{array}{ll}
 d \longrightarrow u + e^- + \bar{\nu}_e & \beta^- \text{ decay} \\
 u \longrightarrow d + e^+ + \nu_e & \beta^+ \text{ decay} \\
 u + e^- \longrightarrow d + \nu_e & \text{electron capture } (\varepsilon)
 \end{array} \quad (3)$$

that represents the transitions of nucleons, as really transitions between the up (u) and down (d) quarks. A particle physics picture of β^- -decay is given in Figure 2.

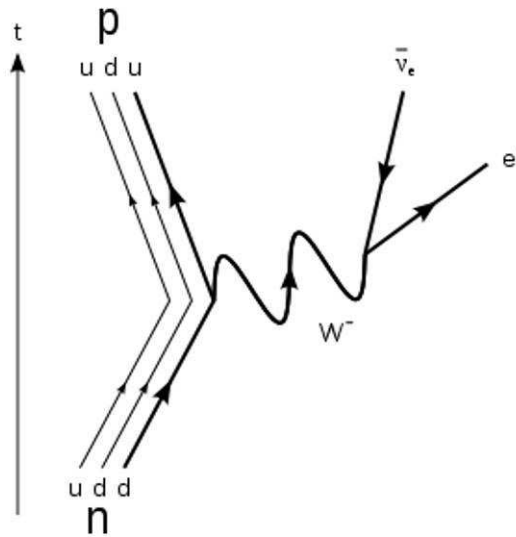


Figure 2: The particle physics view of β^- -decay. In this case, the weak force is carried by the intermediate vector boson, the W^- . In the case of β^+ -decay, the weak force is carried by the intermediate vector boson, the W^+ , the antiparticle to the W^- . There is also a neutral intermediate vector boson, Z^0 , that is responsible for such things as $\nu\nu$ scattering.

Consequences of β -decay's 3-body final state

β^\pm -decay has 3 “bodies” in the final state: the recoil daughter nucleus, the e^\pm , and a neutrino.

Typically, the daughter nucleus (even in the case of free neutron decay, is much more massive than the leptons, therefore, the leptons carry off most of the energy.

Even in the worst possible case, that of free neutron decay, the recoil proton can at most about 0.4 keV, or about 0.05% of the reaction Q -value.

Consequently, if one measures the kinetic energy of the resultant electron, one measures a distribution of energies, that (generally) peaks at small energies, and reaches an “end-point” energy, the so-called β -*endpoint*.

This β -endpoint represents the case where the ν 's energy approaches zero. See Figure 3.

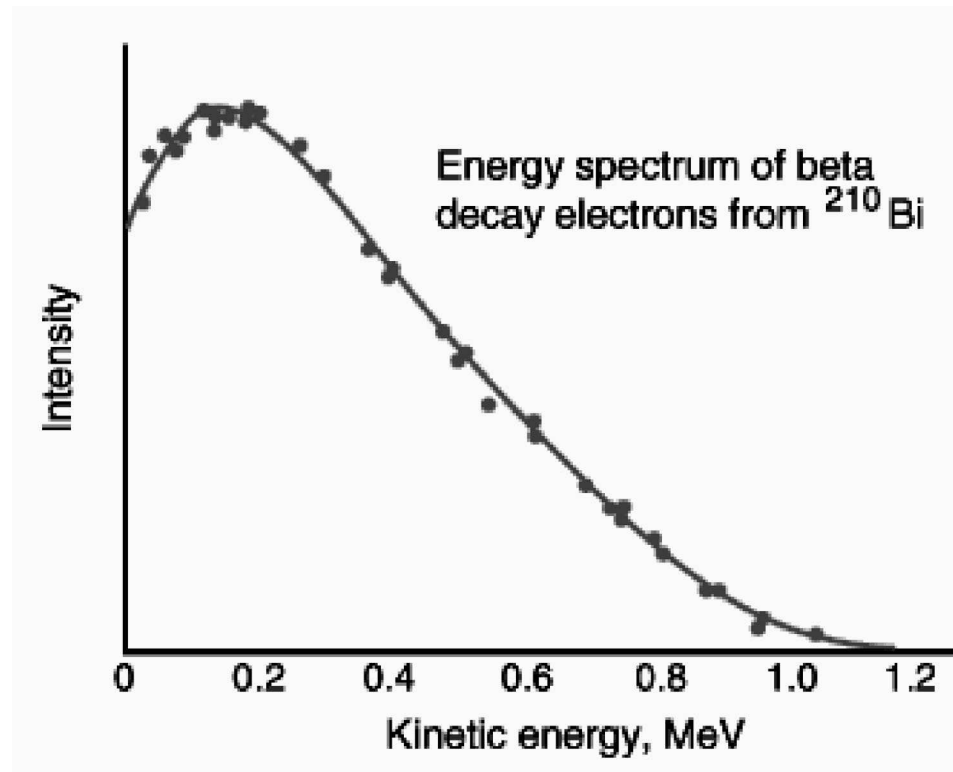


Figure 3: A typical electron energy spectrum that is measured in a β decay. The endpoint energy is the maximum energy that can be given to the electron, and that is closely related to the reaction Q -value (small recoil correction). At lesser energies, the ν carries off some of the available kinetic energy that Q provides.

This leads naturally to a discussion of ...

Chapter 15.1: Energy release in β decay

Neutron decay

$$\begin{aligned}n &\longrightarrow p + e^{-} + \bar{\nu}_e \\m_n c^2 &= m_p c^2 + m_e c^2 + m_{\bar{\nu}_e} c^2 + Q_n \\Q_n &= m_n c^2 - m_p c^2 - m_e c^2 - m_{\bar{\nu}_e} c^2 \\Q_n &= (939.5655133(58) - 938.2720813(58) - 0.5109989461(31))[\text{MeV}] - m_{\bar{\nu}_e} c^2 \\Q_n &= 0.7824331(58)[\text{MeV}] - m_{\bar{\nu}_e} c^2\end{aligned}\tag{4}$$

Since $4 \times 10^{-8} < m_{\bar{\nu}_e} c^2 < 2.2 \times 10^{-6}[\text{MeV}]$, we can safely ignore the neutrino rest mass energy, within the experimental uncertainty of the reaction Q ,

$$Q_n = 0.7824331(58)[\text{MeV}]\tag{5}$$

Accounting for proton recoil, the relativistically exact relationship between the electron endpoint energy and Q , is given by:

$$T_e^{\max} = Q \left[\frac{1 + Q/(2m_p c^2)}{1 + m_e/m_p + Q/m_p c^2} \right]. \quad (6)$$

The above does not consider the loss of energy the electron experiences from attraction by the positive proton. This is likely in the neighborhood of a few eV and can be ignored. (To be verified.)

Putting in numerical values, we calculate $T_e^{\max} = 0.7816817(58)[\text{MeV}]$, which agrees with the direct measurement of $T_e^{\max} = 0.782(13)[\text{MeV}]$.

We can calculate the proton's recoil energy by using Conservation of Energy:

$$T_p^{\max} = Q_n - T_e^{\max} \quad (7)$$

This evaluates numerically to $T_p^{\max} \approx 0.7513(82)[\text{keV}]$ and is corroborated by measurements of the proton recoil spectrum. Data is shown on the next page.

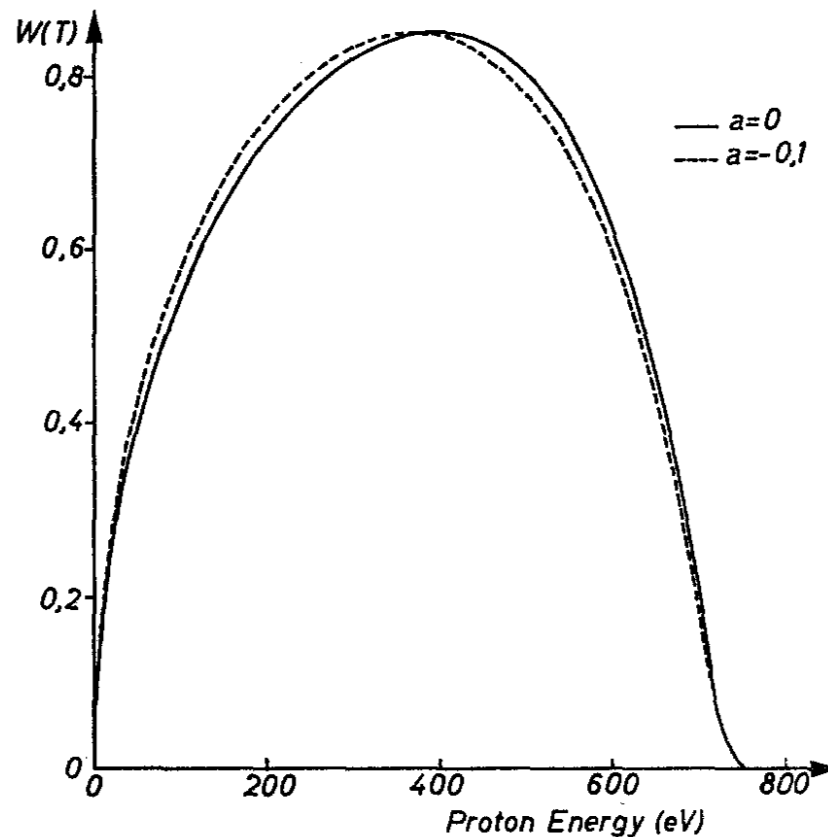


Fig. 2. The expected shape of the recoil proton energy spectrum in the c.m. system of the neutron. The two curves correspond to two assumed values of the angular correlation coefficient. The true value (to be measured by this experiment) is about $a = -0.1$

From O. Nachtmann and H. Paul, "The Proton Recoil Spectrum of Thermal Neutrons Decaying in Vacuum", Z. Physik 226, 17-27 (1960).

Q for β^- -decay

For β^- -decay

$${}^A_ZX_N \longrightarrow {}^A_{Z+1}X'_{N-1} + e^- + \bar{\nu}_e \quad (8)$$

Going back to the definition of Q in terms of nuclear masses, and ignoring, henceforth, the mass of the neutrino:

$$Q_{\beta^-} = [m_N({}^A_ZX_N) - m_N({}^A_{Z+1}X'_{N-1}) - m_e] c^2, \quad (9)$$

where the subscript “N” connotes nuclear (not atomic) masses.

The relationship between the nuclear (no subscript “N”) and atomic mass is:

$$m({}^A_ZX_N)c^2 = m_N({}^A_ZX_N)c^2 + Zm_e c^2 - \sum_{i=1}^Z B_i, \quad (10)$$

where B_i is the binding energy of the i 'th atomic electron.

Substituting (10) in (9), to eliminate the (less well known) nuclear masses results in:

$$\begin{aligned}
 Q_{\beta^-} &= [m({}_Z^AX_N) - Zm_e] c^2 - [m({}_{Z+1}^AX'_{N-1}) - (Z+1)m_e] c^2 - m_e c^2 \\
 &\quad + \left[\sum_{i=1}^Z B_i - \sum_{i=1}^{Z+1} B'_i \right] \\
 &= [m({}_Z^AX_N) - m({}_{Z+1}^AX'_{N-1})] c^2 + \left[\sum_{i=1}^Z B_i - \sum_{i=1}^{Z+1} B'_i \right] , \\
 &= [m({}_Z^AX_N) - m({}_{Z+1}^AX'_{N-1})] c^2 + \left[\sum_{i=1}^Z (B_i - B'_i) - B'_{Z+1} \right] , \tag{11}
 \end{aligned}$$

noting that the electron masses have canceled in this case. The factor

$$\sum_{i=1}^Z B_i - \sum_{i=1}^{Z+1} B'_i = \sum_{i=1}^Z (B_i - B'_i) - B'_{Z+1}$$

is the difference in the energy of the electronic orbital configuration of the parent and daughter nuclei. Generally, this difference can be ignored. However, in the case of large Z nuclei, it can amount to about 10 keV. For accurate determinations of Q , the difference in atomic electron binding energy must be accounted for.

Q for β^+ -decay

Similar considerations for β^+ decay lead to:

$$Q_{\beta^+} = \left[m({}_Z^A X_N) - m({}_{Z-1}^A X'_{N+1}) - 2m_e \right] c^2 + \left[\sum_{i=1}^Z B_i - \sum_{i=1}^{Z-1} B'_i \right] . \quad (12)$$

Here we note that the electron rest-mass energies do not completely cancel. However, the discussion regarding the electron binding energy remains the same.

Q for electron capture

For electron capture:

$$Q_{\varepsilon} = [m({}_Z^A X_N) - m({}_{Z-1}^A X'_{N+1})] c^2 - B_n + \left[\sum_{i=1}^Z B_i - \sum_{i=1}^{Z-1} B'_i \right] . \quad (13)$$

The latter term related to electron binding energy,

$$\sum_{i=1}^Z B_i - \sum_{i=1}^{Z-1} B'_i$$

is generally ignored, for the reasons cited above. However, the the binding energy of the captured electron, B_n can approach 100 keV for large- Z nuclei, and can *not* be ignored.

Discussion point: Free neutron decay, revisited

From our current understanding of the weak interaction, the electron is created when a down quark changes into an up quark. The Q value for this reaction is 0.782 MeV.

Let us see if we can apply some reasoning from classical physics to say something about the observation of such a decay.

If the electron were a “point” particle, and it was created somewhere inside the neutron at radius r , it would feel an attraction:

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \left\{ \frac{\Theta(R_p - r)}{R_p} \left[\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R_p} \right)^2 \right] + \frac{\Theta(r - R_p)}{r} \right\} ,$$

where R_p is the radius of the proton.

We are assuming that the quarks are moving so fast inside the proton, that all the electron sees is a continuous blur of charge adding up to one unit of charge.

So, if $R_p \approx 1.2$ fm (from $R_N = (1.22 - 1.25)[\text{fm}]A^{1/3}$), we can conclude that the kinetic energy that the electron is required to have to escaped the nucleus falls in the range:

$$\frac{e^2}{4\pi\epsilon_0 R_p} \leq T_e \leq \frac{3}{2} \frac{e^2}{4\pi\epsilon_0 R_p} .$$

Evaluating:

$$1.2 \text{ [MeV]} \leq T_e \leq 1.8 \text{ [MeV]} > Q = 0.782$$

In other words, it can not happen, since there is a contradiction with the observation that it does decay, with a meanlife of about 11 minutes, with the given Q .

How do we explain this?

We can only conclude that we have observed a fundamentally new phenomenon.

Chapter 15.2: Fermi's theory of β decay

Fermi's theory of β decay starts with a statement of Fermi's Golden Rule #2 for transition rate, λ :

$$\lambda = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E_{if}) , \quad (14)$$

where V is a potential that causes the transition from an initial quantum state Ψ_i (the parent nucleus in this case) to a final one, Ψ_f , that includes wavefunctions of the daughter nucleus, the electron and its neutrino. $V_{if} \equiv \langle \Psi_f | V | \Psi_i \rangle$ is the transition amplitude.

The derivation of Fermi's Golden Rule #2 is generally reserved for graduate courses in Quantum Mechanics, but a version of the derivation is available in Chapter 13, for your interest.

What concerns us now, is to calculate the density of final states, $\rho(E_{if})$, for the β -transition. This derivation figures prominently in the β -spectrum, and the endpoint energy.

Starting in Chapter 13, the density of states is derived for non-relativistic particles with mass, relativistic particles with mass (the electron in this case), and massless particles (the neutrino in this case).

We start with (13.21). The number of states, N , of a particle in the final state with energy E is given by:

$$dN = \frac{\pi}{2} n^2 dn . \quad (15)$$

where $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$, and (n_x, n_y, n_z) are the quantum numbers of a free particle in an infinite box potential, with side L . the momentum and the n 's are related by:

$$p_i = n_i \pi \hbar / L . \quad (16)$$

Putting (16) into (15) gives:

$$dN = \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 dp . \quad (17)$$

Or, dividing by dE ,

$$\frac{dN}{dE} = \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 \frac{dp}{dE} . \quad (18)$$

We should point out that (18) is valid for all particles, massless, relativistic and non-relativistic, since (16) is universal.

All we need do now is relate momentum to energy to compute the density factors. For the neutrino, which we are now treating as massless,

$$\begin{aligned}
 p_\nu &= E_\nu/c \\
 dp_\nu &= dE_\nu/c \\
 \frac{dN_\nu}{dE_\nu} &= \frac{1}{2\pi^2} \frac{L^3}{\hbar^3 c^3} E_\nu^2
 \end{aligned} \tag{19}$$

For the electron, that must be treated relativistically,

$$\begin{aligned}
 p_e &= \sqrt{E_e^2 - (m_e c^2)^2}/c \\
 dp_e &= [E_e/(c\sqrt{E_e^2 - (m_e c^2)^2})]dE_e \\
 \frac{dN_e}{dE_e} &= \frac{1}{2\pi^2} \frac{L^3}{\hbar^3 c^3} \sqrt{E_e^2 - (m_e c^2)^2} E_e \\
 \frac{dN_e}{dT_e} &= \frac{1}{2\pi^2} \frac{L^3}{\hbar^3 c^3} \sqrt{T_e(T_e + 2m_e c^2)}(T_e + m_e c^2)
 \end{aligned} \tag{20}$$

For β decay we have two particles in the final state, so we can express the rate of decay to produce an electron with momentum p as:

$$\frac{d\lambda_\beta}{dp} = \frac{2\pi}{\hbar} |V_{if}|^2 \frac{dN_e}{dp} \frac{dN_\nu}{dE_{if}} , \quad (21)$$

If q is the momentum of the neutrino,

$$\begin{aligned} E_{if} &= T_e + cq \\ dE_{if} &= c(dq) \quad (T_e \text{ fixed}) . \end{aligned} \quad (22)$$

Thus,

$$\frac{d\lambda_\beta}{dp} = \frac{2\pi}{\hbar c} |V_{if}|^2 \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} p^2 \frac{1}{2\pi^2} \frac{L^3}{\hbar^3} q^2 \delta(E_{if} - [T_e + T_\nu]) . \quad (23)$$

Where the δ -function accounts specifically for the conservation of energy.

Recall that the free electron and neutrino wavefunctions have the form $L^{-3/2} \exp(i\vec{p} \cdot \vec{x} / \hbar)$ and $L^{-3/2} \exp(i\vec{q} \cdot \vec{x} / \hbar)$, respectively.

Thus, the L for the side of the box disappears from the calculation. We also replace $q = (Q - T_e)/c$, ignoring the recoil of the daughter nucleus. Finally, integrating over all possible neutrino energies, we obtain:

$$\begin{aligned} \frac{d\lambda_\beta}{dp} &= \frac{|\mathcal{M}_{if}|^2}{2\pi^3 \hbar^7 c^3} p^2 (Q - T_e)^2 \text{ or} \\ \frac{d\lambda_\beta}{dp} &= \frac{|\mathcal{M}_{if}|^2}{2\pi^3 \hbar^7 c} p^2 q^2 \end{aligned} \quad (24)$$

where $\mathcal{M}_{if} = L^3 V_{if}$.

Thus we have derived Fermi's celebrated equation.

Just a brief note on dimensions:

$|V_{if}|^2$ has units $[E^2]$ because all the wavefunctions inside are normalized.

Getting rid of all the L 's results in \mathcal{M}_{if} having units $[\text{length}^3 \times \text{energy}]$. (24) *is* correct dimensionally.

Allowed transitions

Now we examine the form of the “matrix element” \mathcal{M}_{if} .

This has changed form several times during the derivation, and will again, to conform with Krane’s book.

We now rewrite

$$\begin{aligned}\mathcal{M}_{if} &= gM_{if} \\ M_{if} &= \langle (e^{i\vec{p}\cdot x/\hbar})(e^{i\vec{q}\cdot x/\hbar})\psi_{X'} | \mathcal{O}_\beta | \psi_X \rangle ,\end{aligned}\tag{25}$$

where g is the “strength” of the β transition. From experiments, it is known that:

$$g \approx 0.88 \times 10^{-4} \text{ MeV fm}^3 .$$

This g is a scalar quantity that plays the role of e , the electric charge, for electromagnetic transitions. The unnormalized electron wavefunction is $\exp(i\vec{p}\cdot x/\hbar)$, and the unnormalized neutrino wavefunction is $\exp(i\vec{q}\cdot x/\hbar)$. $\psi_{X'}$ is the wavefunction of the daughter nucleus, while ψ_X is the wavefunction of the parent nucleus.

Finally, \mathcal{O}_β is the weak interaction operator, the cause of the transition.

We recall from the class discussions, that the electron and neutrino wavefunctions have wavelengths that are many times the size of the nucleus.

So, it seems reasonable to expand these wavefunctions in a Taylor series expansion, to see how far we get. Namely,

$$\begin{aligned}\exp(i\vec{p} \cdot x/\hbar) &= 1 + \frac{i\vec{p} \cdot x}{\hbar} - \left(\frac{\vec{p} \cdot x}{\hbar}\right)^2 + \dots \\ \exp(i\vec{q} \cdot x/\hbar) &= 1 + \frac{i\vec{q} \cdot x}{\hbar} - \left(\frac{\vec{q} \cdot x}{\hbar}\right)^2 + \dots\end{aligned}\tag{26}$$

Thus the leading-order term of (25) is:

$$M_{if}^0 = \langle \psi_{x'} | \mathcal{O}_\beta | \psi_x \rangle .\tag{27}$$

If $M_{if}^0 \neq 0$, the β decay is called an “allowed” transition, and the rate is relatively prompt.

If $M_{if}^0 = 0$, then we must go to higher order terms in (26). These are called “forbidden” transitions, and occur, but at much slower rates. (More on this topic later.)

Krane likes to adopt the following shorthand. For allowed transitions, we see that:

$$\frac{d\lambda_{\beta}^0}{dp} = g^2 \frac{|M_{if}^0|^2}{2\pi^3 \hbar^7 c} p^2 q^2 . \quad (28)$$

If we have $N(t)$ β -emitters in a sample, the momentum spectrum of electrons that may be measured is:

$$N^0(p)dp = N(t)d\lambda_{\beta}^0 = \left(g^2 N(t) \frac{|M_{if}^0|^2}{2\pi^3 \hbar^7 c^5} \right) p^2 q^2 dp . \quad (29)$$

If $N(t)$ changes little over the course of the measurement of the spectrum (This is the usual case.):

$$N^0(p)dp = C^{(0)} p^2 q^2 dp , \quad (30)$$

where we have gathered all constants with inside the large parentheses in (29) into a global constant $C^{(0)}$, that is determined experimentally. It can be determined through the a normalization condition,

$$\int dp N^0(p) \equiv 1 .$$

Conventional forms: $N^0(p)$, $N^0(T_e)$

$N^0(p)$ expressed in (30) contains p and q , that are related by conservation of energy. In terms of single momentum variable,

$$N^0(p)dp = \frac{C^{(0)}}{c^2} p^2 \left[Q - [\sqrt{(cp)^2 + (m_e c^2)^2} - m_e c^2] \right]^2 dp, \quad (31)$$

since

$$cq = Q - T_e = Q - \{[\sqrt{(cp)^2 + (m_e c^2)^2} - m_e c^2]\}$$

The maximum possible p occurs when the neutrino component drops to zero. This is easily found to be:

$$p_{\max} = \frac{1}{c} \sqrt{Q^2 + 2Qm_e c^2} \quad (32)$$

by setting cq above to zero.

An even more common expression is to show N^0 in terms of T_e .

We find this by saying:

$$N^0(T_e)dT_e = N^0(p)dp = N^0(p) \left(\frac{dp}{dT_e} \right) dT_e , \quad (33)$$

Applying relativistic kinematic relationships, we find:

$$N^0(T_e)dT_e = \frac{C^{(0)}}{c^5} \sqrt{T_e^2 + 2T_em_ec^2}(T_e + m_ec^2)(Q - T_e)^2 dT_e . \quad (34)$$

Here the β -endpoint at $Q = T_e$ is evident.

Not also, the parabolic shape, *viz.* $(Q - T_e)^2$.

Look for this in plots of the β spectrum, when it is expressed in the form of (34).

Accounting of "forbiddenness" and nuclear Coulomb effect

There are two other attributes of β -spectra we must take account of, before we start using the theoretical spectral shape to assist in analyzing data.

The Nuclear Coulomb Effect ...

accounts for the interaction of the daughter's Coulomb charge with the resultant electron or positron in the final state.

This nuclear charge has no effect on the neutral neutrino.

In (25), we wrote the electron wavefunction as a free plane wave. In actual fact, that was a fairly crude approximation.

These plane waves are distorted significantly by the attraction the β^- would feel, and the repulsion that the positron would feel.

Accounting for this is quite involved, but not beyond our capabilities.

We would have to go back to (25) and write the electron wave functions in terms of free particle solutions to the Coulomb potential, using a relativistic analysis. (That would be high-level graduate course.)

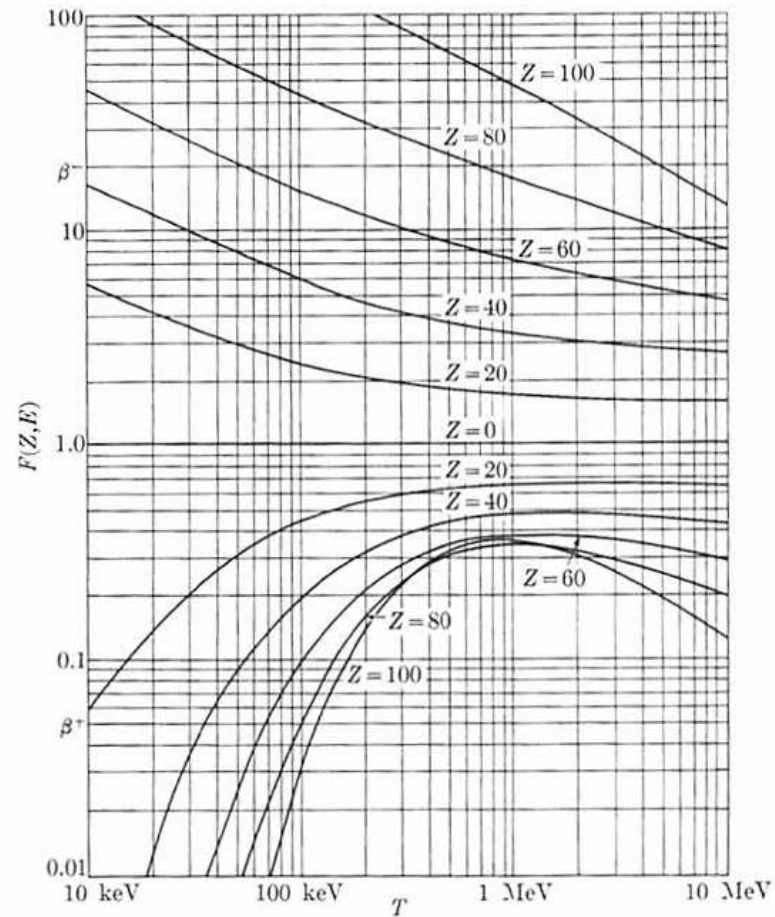
However, I have never seen detailed discussion of this Coulomb factor in even graduate-level texts, and interested students are usually told to seek out the papers in the literature.

The result is, however, that the β -spectra are multiplied by a correction factor, the *Fermi function*, that depends on the charge of the daughter nucleus, Z' , and the electron momentum and sign, $F^\pm(Z', p)$. The effect it has could have been anticipated from classical considerations. The electron spectra is dragged back toward lesser values, while the positron spectra are pushed toward higher values.

The mathematical form is:

$$F^\pm(Z', p) = 2(1 + \kappa_0)(2pR_N/\hbar)^{-2(1-\kappa_0)} \exp(\pi\nu) \frac{|\Gamma(\kappa_0 + i\nu)|^2}{\Gamma(2\kappa_0 + 1)^2},$$

where $\Gamma()$ is the Gamma function, R_N is the nuclear radius (assumed to be a uniform sphere of charge), $\kappa_0 = \sqrt{1 - (\alpha Z')^2}$, α is the fine-structure constant, $\nu = \pm Z'e^2/(4\pi\epsilon_0\hbar v)$, for $\mp e$, and v is the electron velocity. See the figures on the next two pages.



The Coulomb factor (Fermi function) $F(Z, E)$ plotted vs. T , the kinetic energy of the electron or positron. (Prepared from *Tables for the Analysis of Beta Spectra*, National Bureau of Standards, Applied Mathematics Series, 13.)

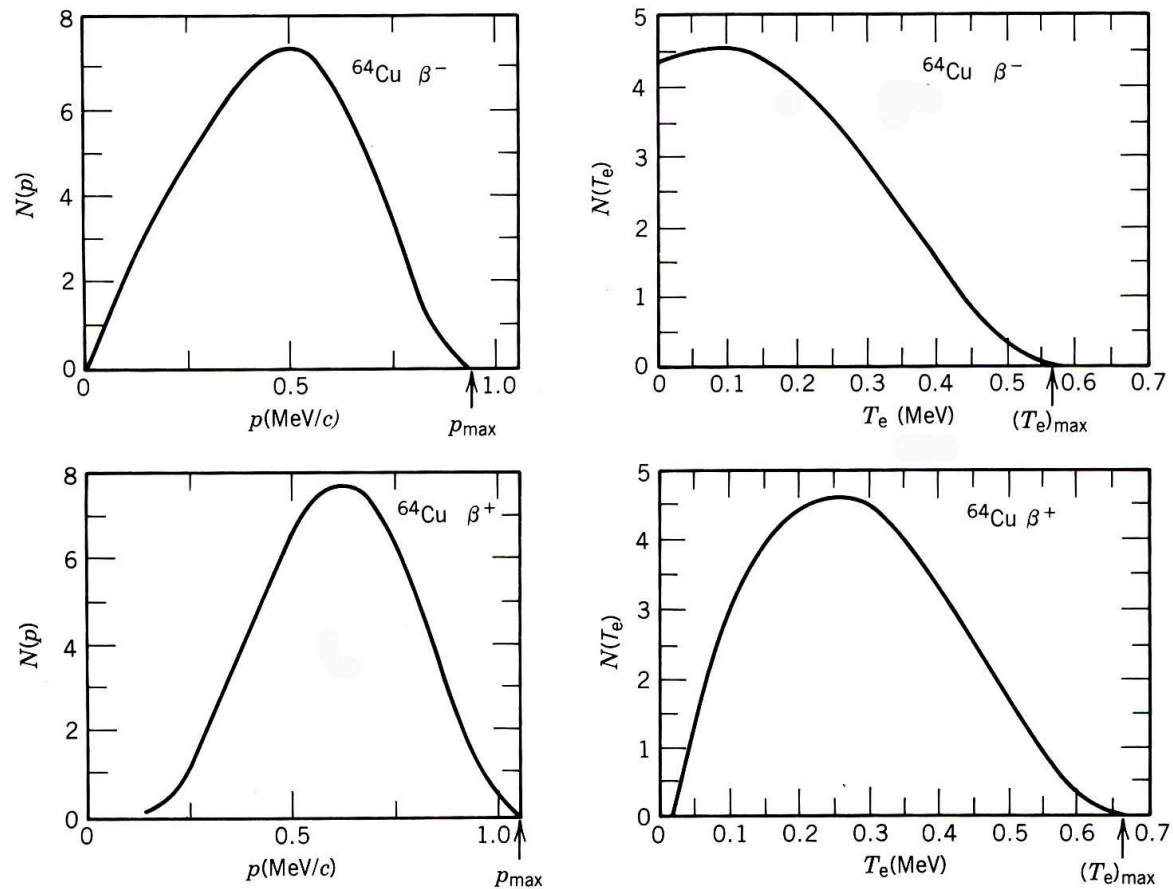


Figure 9.3 Momentum and kinetic energy spectra of electrons and positrons emitted in the decay of ^{64}Cu . Compare with Figure 9.2; the differences arise from the Coulomb interactions with the daughter nucleus. From R. D. Evans, *The Atomic Nucleus* (New York: McGraw-Hill, 1955).

In the determination of total decay rates, the entire β spectrum is integrated, in order to count the number of β decay's in a given time interval, to extract the decay rate.

An integrated form of the Fermi function appears in that case,

$$f(Z', E_0) = \int_0^{p_{\max}} \frac{p^2 dp}{(m_e c)^3} \frac{(E_0 - E_e)^2}{(m_e c^2)^2} F(Z', p) ,$$

that is explicitly dimensionless, by design.

Here, p is the electron's momentum, p_{\max} is the β endpoint in terms of the electron's momentum, E_e is the electron energy, E_0 is the β endpoint in terms of the electron's energy, and $F(Z', p)$ is the Fermi function, as seen in the figure, two pages back.

The integrated Fermi function, $f(Z', E_0)$, is graphed on the following page.

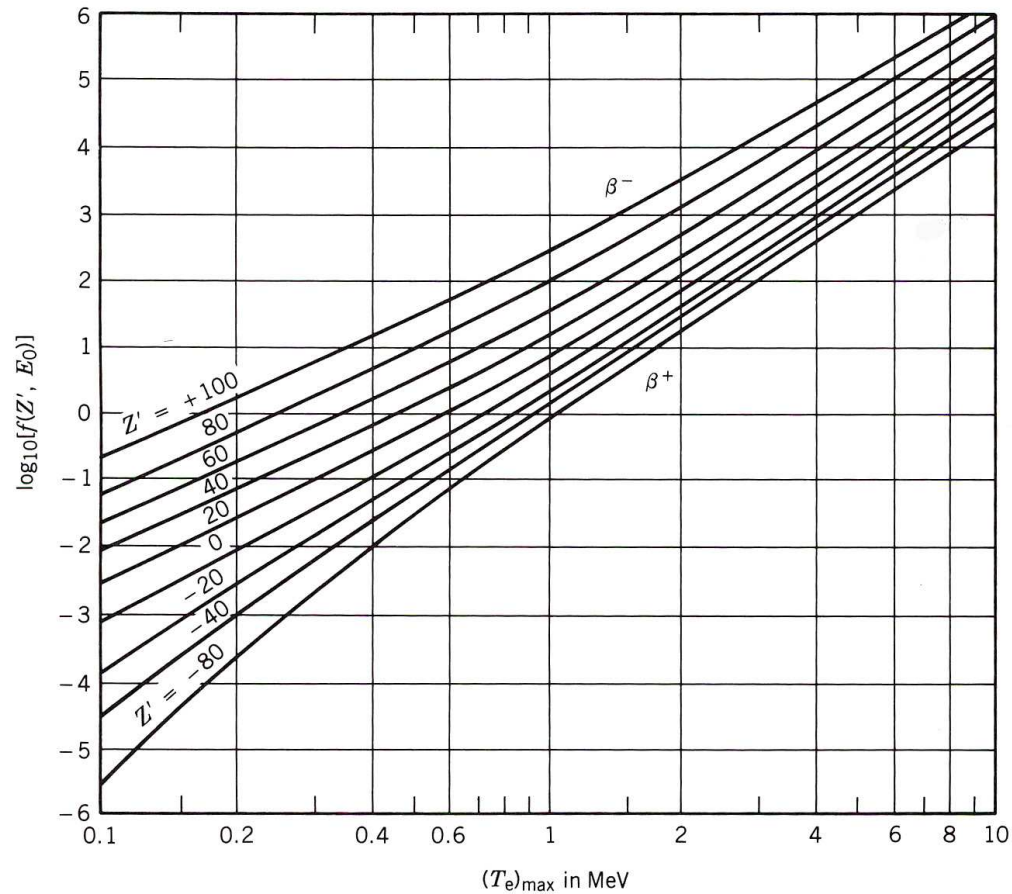


Figure 9.8 The Fermi integral, defined by Equation 9.29. The atomic number Z' refers to the daughter nucleus; the curves for positive Z' are for β^- decay, while negative Z' is for β^+ decay. From R. D. Evans, *The Atomic Nucleus* (New York: McGraw-Hill, 1955).

Degree of “forbiddenness”

The “forbiddenness” of the decay also affects the shape of the spectrum. This is also a multiplicative correction to the β -spectrum.

There are difference shapes depending on the level of “forbiddenness”, and that is determined by the amount of orbital angular momentum, L , carried away by the electron-neutrino pair, as well as their momenta.

Examples of these shape factors are given in the table on the next page, for the “unique forbidden transitions”⁴.

The mathematical form of the shape factor, for the unique forbidden transitions, is:

$$S^L(p, q) = \frac{\int_{-1}^1 d\mu (p^2 + q^2 + 2pq\mu)^L}{2(m_e c)^{2L}}$$

⁴Relativistic quantum mechanics allows us to calculate these in the special case of unique transitions. These transitions are ones in which the angular momentum vector and the two lepton spins are all aligned.

L	$S^L(p, q)$	
0	1	Allowed
1	$(p^2 + q^2)/(m_e c)^2$	Unique first forbidden
2	$(p^4 + \frac{10}{3}p^2q^2 + q^4)/(m_e c)^4$	Unique second forbidden
3	$(p^6 + 7p^4q^2 + 7p^2q^4 + q^6)/(m_e c)^6$	Unique third forbidden
\vdots	\vdots	\vdots

Table 1: Shape factors for the first three unique forbidden transitions.

The β -spectrum revealed

With all these various factors affecting the spectral shape and decay rates for β decay, we write down the final form that is employed for data analysis:

$$N(p) \propto |M_{if}^L|^2 p^2 (Q - T_e)^2 S^L(p, q) F^\pm(Z', p) , \quad (35)$$

where,

1. M_{if}^L is the nuclear matrix element associate with the transition. It can depend on p and q , as well as the alignment of spin and angular momentum vectors. It exhibits a very strong dependence on the angular momentum, L , carried off by the lepton pair. M_{if}^L also depends strongly on the “closeness” of the initial and final nuclear quantum wavefunctions. The closer the initial and final nuclear quantum states are, the larger their overlap, resulting in a larger M_{if}^L .
2. $p^2(Q - T_e)^2$ is the “statistical factor” associated with the density of final states.
3. $F^\pm(Z', p)$, the Fermi function. It accounts for the distortion of the spectral shape due to attraction/repulsion of the electron/positron.
4. $S^L(p, q)$ accounts for spectral shape differences. It depends on the total orbital angular momentum carried off by the electron-neutrino pair, \vec{L} , their total spin value, \vec{S} , and their orientation with respect to each other.

Chapter 15.3: Experimental tests of Fermi's theory

Kurie plots: Shape of the β spectrum

To employ (35) to analyze β spectra, one plots:

$$\sqrt{\frac{N(p)}{S^L(p, q)F^\pm(Z', p)}} \quad vs. \quad T_e, \quad (36)$$

using the initial assumption that $L = 0$, so that $S^L(p, q) = 1$.

If the data points fall on a straight line (statistical tests may be necessary), one can easily obtain the Q -value from the abscissa axis from the intercept where the ordinate is zero.

This type of plot is called a Kurie plot (named after Franz N. D. Kurie, who published a paper, with two co-authors [J. R. Richardson and H. C. Paxton], on β -spectrum analysis). If the line is straight, one has also identified, from its shape, that this is an allowed transition.

An example of a Kurie plot, for an allowed transition, *i.e.* $L_{e\nu} = 0$, is shown on the next page.

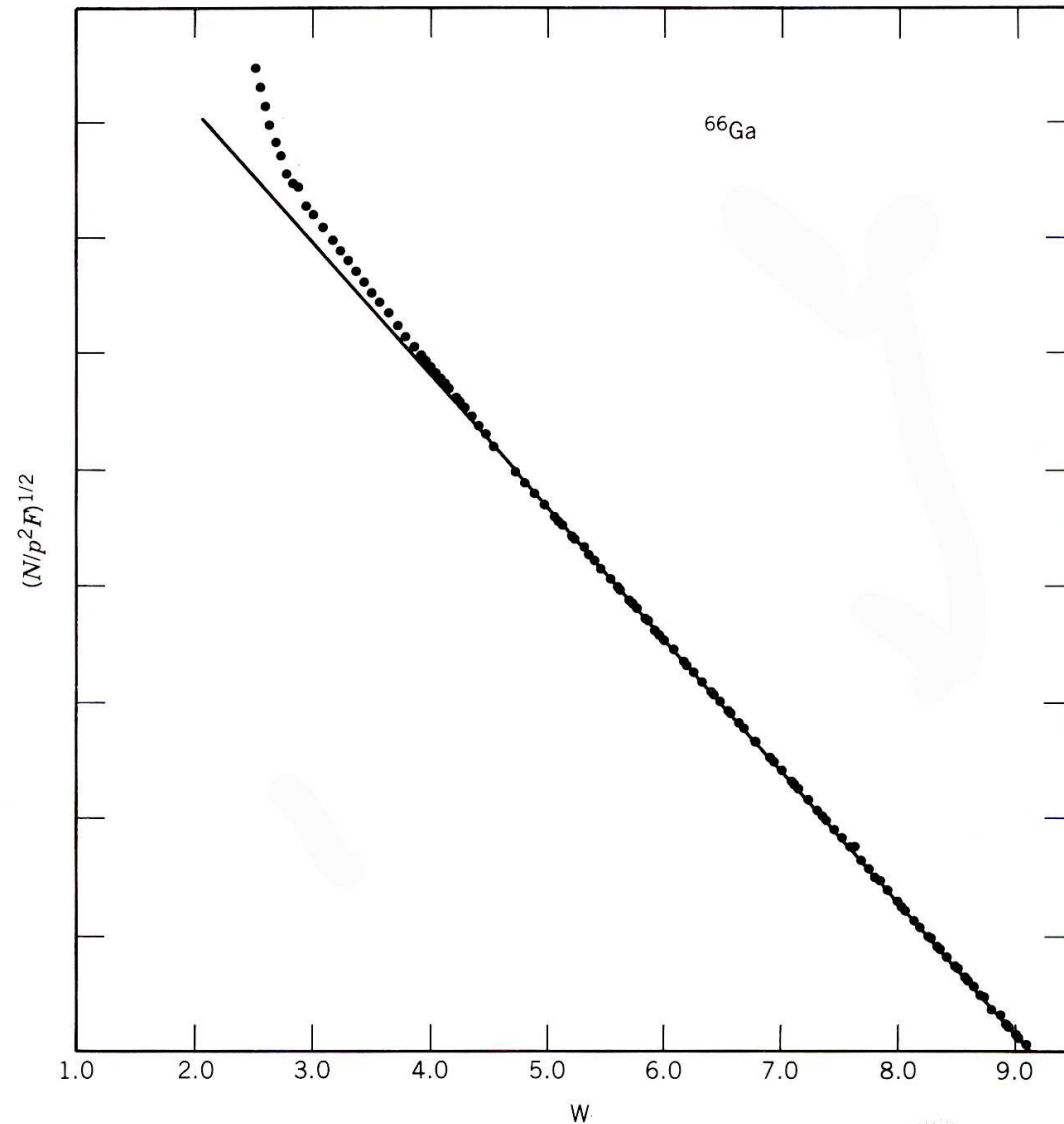


Figure 9.4 Fermi–Kurie plot of allowed $0^+ \rightarrow 0^+$ decay of ^{66}Ga . The horizontal scale is the relativistic *total* energy ($T_e + m_e c^2$) in units of $m_e c^2$. The deviation from the straight line at low energy arises from the scattering of low-energy electrons within the radioactive source. From D. C. Camp and L. M. Langer, *Phys. Rev.* **129**, 1782 (1963).

If the Kurie plot is not straight, one must successively test shape factors until a straight line match is obtained.

Once the shape factor is determined, the level of forbiddenness is determined, and the Q -value may be extrapolated from the data unambiguously.

There are several examples on the next few pages.

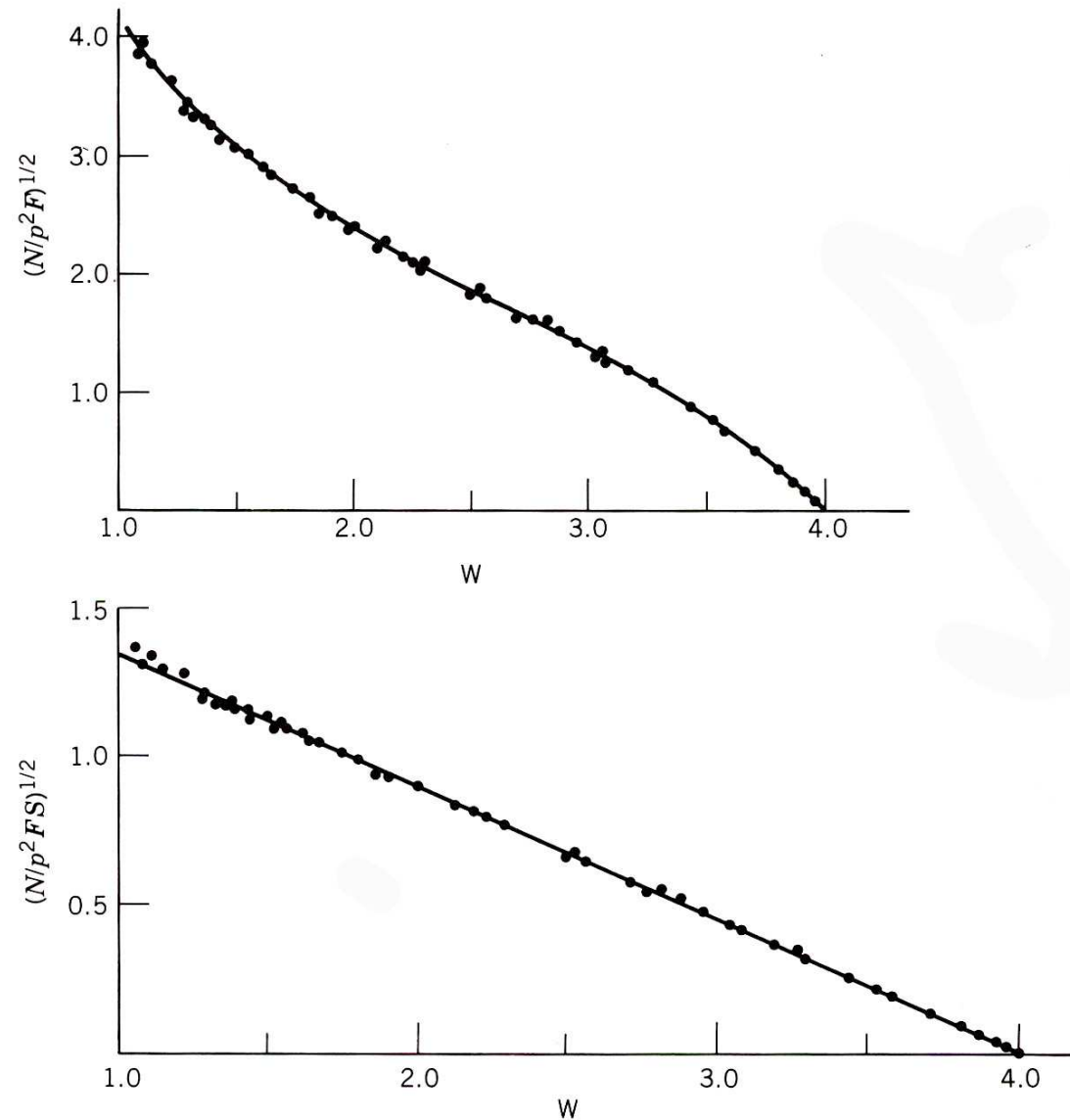


Figure 9.5 Uncorrected Fermi–Kurie plot in the β decay of ^{91}Y (top). The linearity is restored if the shape factor $S(p, q)$ is included; for this type of first-forbidden decay, the shape factor $p^2 + q^2$ gives a linear plot (bottom). Data from L. M. Langer and H. C. Price, *Phys. Rev.* **75**, 1109 (1949).

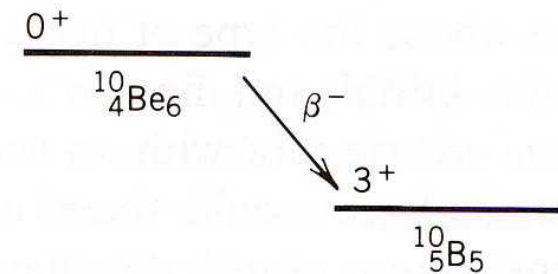
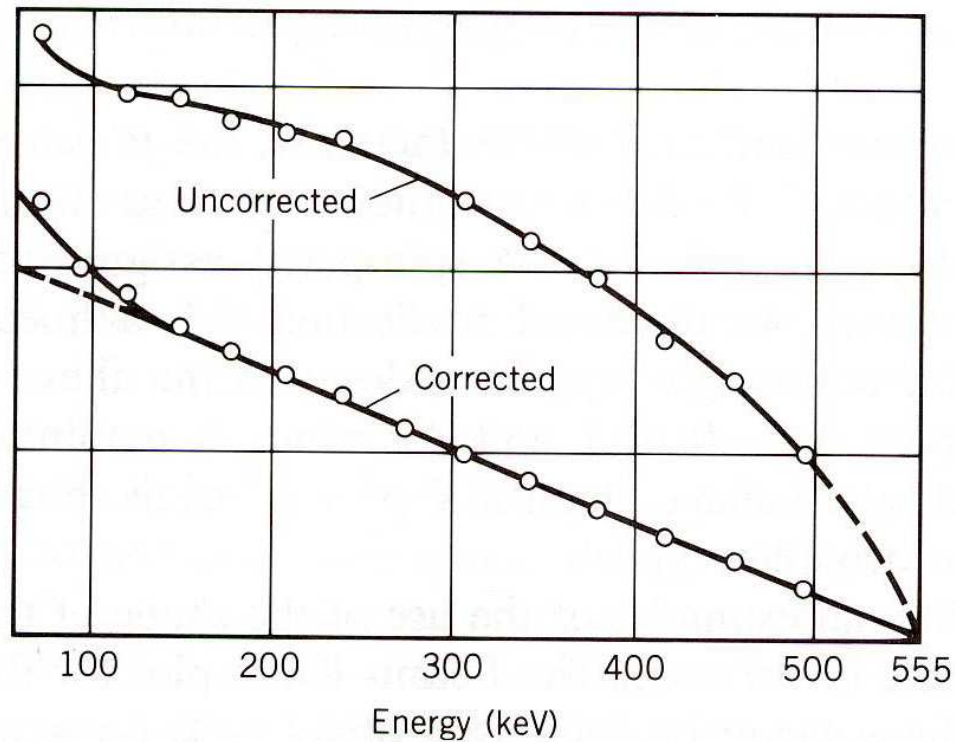


Figure 9.25 Uncorrected Fermi–Kurie plot for ^{10}Be decay and correction for shape factor for second-forbidden transition. Data from L. Feldman and C. S. Wu, *Phys. Rev.* **87**, 1091 (1952).

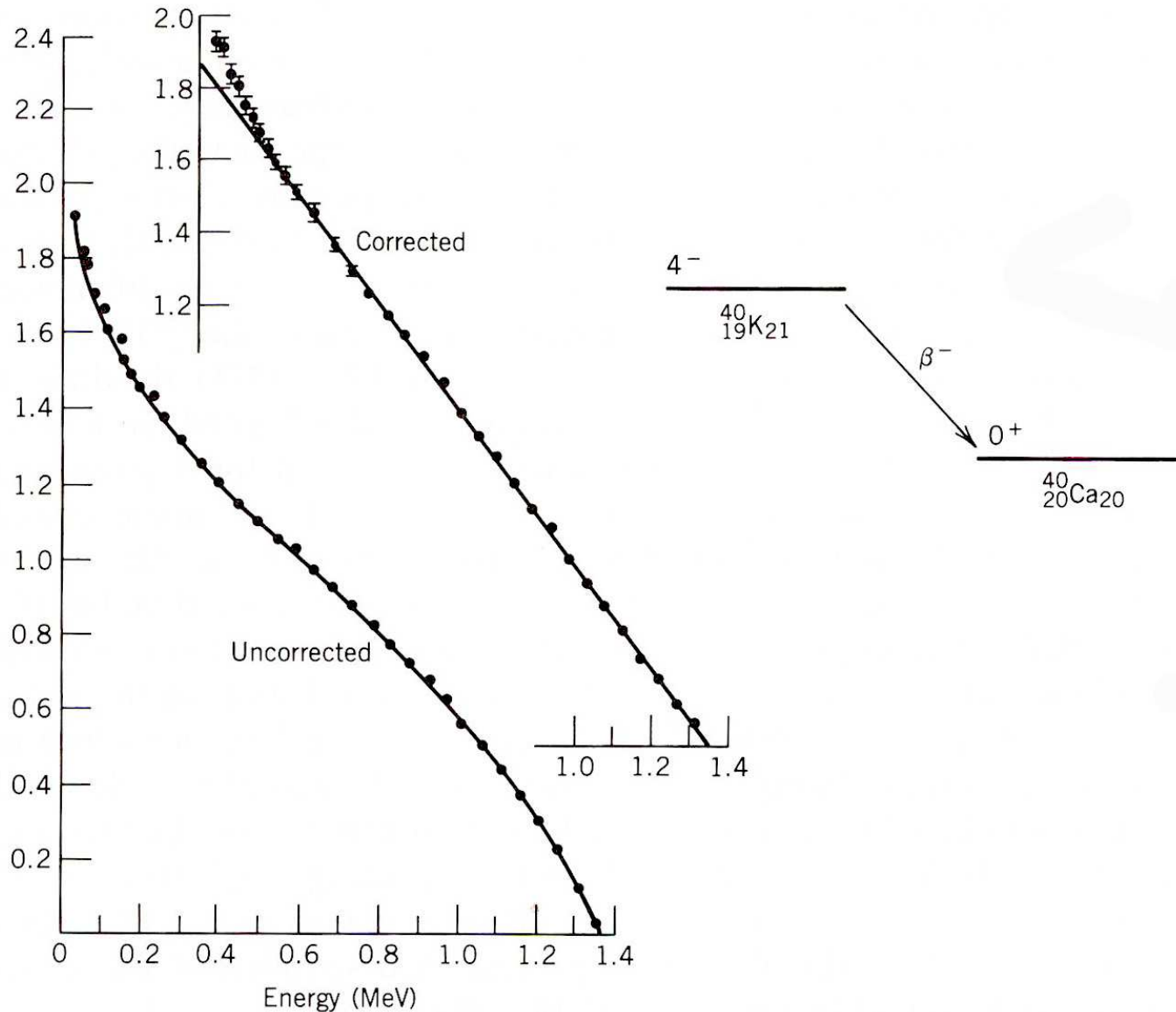


Figure 9.26 Uncorrected Fermi–Kurie plot for the decay of ^{40}K and correction for shape factor of third-forbidden transition. Data from W. H. Kelly et al., *Nucl. Phys.* **11**, 492 (1959).

Unfolding multiple β spectra

Consider the measured β -spectrum shown on the next page, for the decay $^{177}\text{Lu} \longrightarrow ^{177}\text{Hf}$.

These ft of these decays, and their general shape suggests that all the decays are “allowed” transition.

Hence, one can successively subtract off the upper single decay spectra and reveal, in this example, four different endpoint energies.

This particular example is a simple case of well-separated β endpoints, and allowed transitions. For endpoint energies closer together, and degrees of forbiddenness, the extraction would be more complicated, and could be impossible.

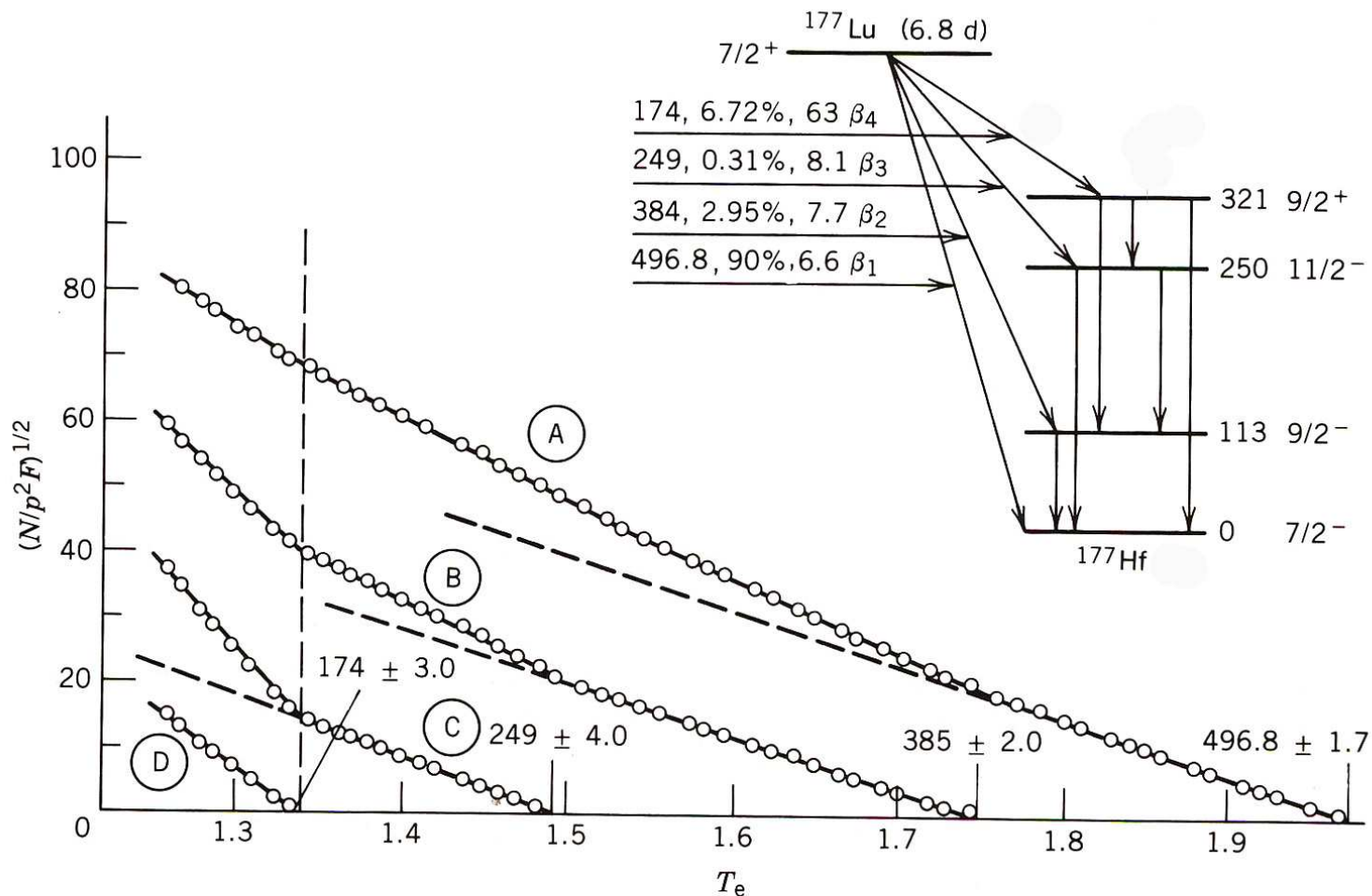


Figure 9.29 Fermi–Kurie plot of the β decay of ^{177}Lu . Curve A represents the complete β spectrum. Extrapolating the high-energy portion (which presumably includes only a single component) gives the dashed line, and the difference between the extrapolated line and curve A gives curve B. The linear portion of curve B gives the endpoint of the next component, and repeating the procedure gives curves C and D. The resulting decay scheme is shown in the inset. Data from M. S. El-Nesr and E. Bashandy, *Nucl. Phys.* **31**, 128 (1962).

Total decay rate: The $ft_{1/2}$, $\log_{10} ft$ values

Putting in the Coulomb and shape factors into (28) allows us to determine the total decay rate for a β -decay process,

$$\begin{aligned}\lambda_\beta &= g^2 \frac{|M_{if}^L|^2}{2\pi^3 \hbar^7 c} \int_0^{p_{\max}} dp S^L(p, q) F^\pm(Z', p) p^2 q^2 \\ &= g^2 \frac{m_e^5 c^4 |M_{if}^L|^2}{2\pi^3 \hbar^7} \left[\frac{1}{(m_e c)^5} \int_0^{p_{\max}} dp S^L(p, q) F^\pm(Z', p) p^2 q^2 \right] \\ &\equiv g^2 \frac{m_e^5 c^4 |M_{if}^L|^2}{2\pi^3 \hbar^7} f_L(Z', Q) ,\end{aligned}\tag{37}$$

where the dimensionless integral in large square brackets, is a theoretical factor that may be pre-computed and employed in the data analysis.

This is conventionally written in terms of half-life, $t_{1/2} = \log(2)/\lambda_\beta$. Thus,

$$f_L(Z', Q) t_{1/2} \equiv ft_{1/2} = \frac{\log_e(2) 2\pi^3 \hbar^7}{g^2 m_e^5 c^4 |M_{if}^L|^2} \approx \frac{6200[\text{s}]}{|M_{if}^L|^2} .\tag{38}$$

This is known colloquially as the ft value. (Pronounced *eff tee*.) The ft 's can be quite large, and sometimes the “ $\log ft$ ” value is quoted. (Pronounced *log eff tee*.) The precise definition is $\log_{10}(ft_{1/2})$.

Mass of the neutrino

Our applications of β -decay ignore the neutrino mass, but they turn out to be critically important for cosmology.

There is one important fact: they **do** have mass, but are small, very small in the case of the e .

The table below shows the current state of the mass determinations of the three generations of leptons, e , μ , and τ .

lepton flavor	neutrino symbol	mass (eV)
e	ν_e	$0.04 \rightarrow 2.2$
μ	ν_μ	$< 1.70 \times 10^5$
τ	ν_τ	$< 1.55 \times 10^7$

Source: http://en.wikipedia.org/wiki/Neutrino_mass#Mass

See also: http://en.wikipedia.org/wiki/Neutrino_oscillation

The upper limit on m_{ν_e} is determined by very careful end-point energy measurements. If the neutrino has mass, the shape of the spectrum at the endpoint goes from having zero slope, to infinite slope.

See the figure on the next page.

The lower bound is measured by observing the neutrino oscillations first achieved in the Super-Kamioka Neutrino Detection Experiment (SK).

SK is a neutrino observatory which is under Mount Kamioka near the city of Hida, Japan.

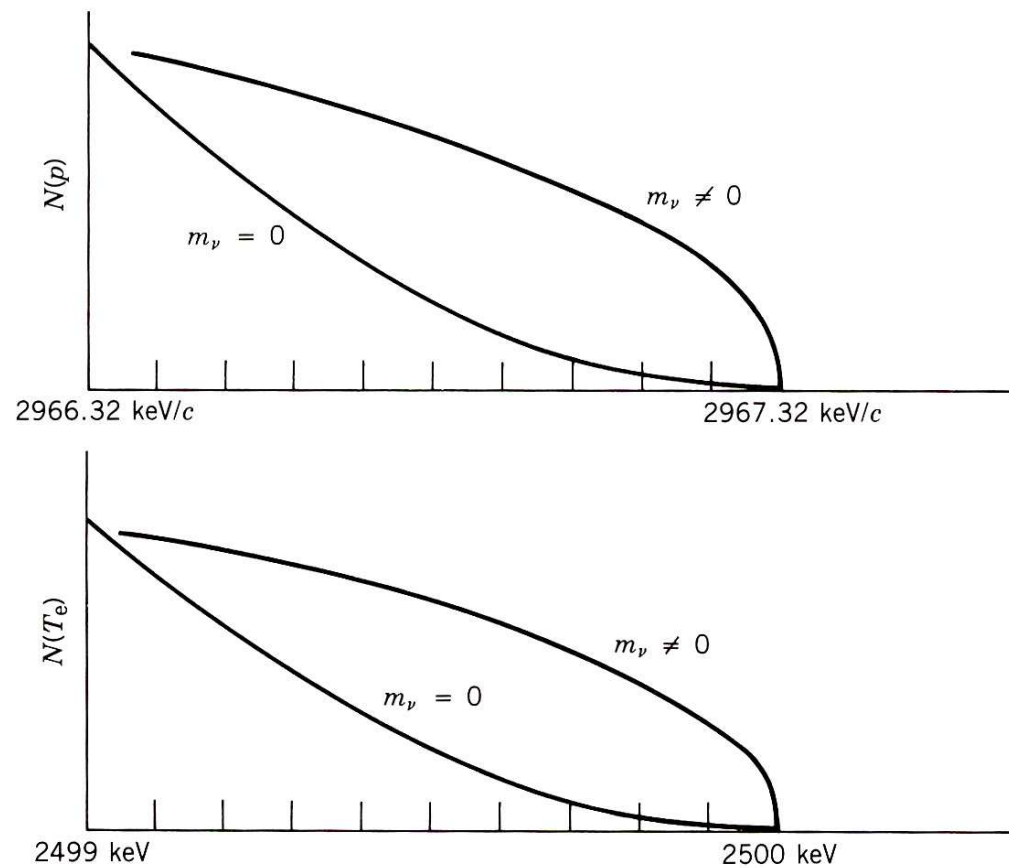


Figure 9.6 Expanded view of the upper 1-keV region of the momentum and energy spectra of Figure 9.2. The normalizations are arbitrary; what is significant is the difference in the shape of the spectra for $m_\nu = 0$ and $m_\nu \neq 0$. For $m_\nu = 0$, the slope goes to zero at the endpoint; for $m_\nu \neq 0$, the slope at the endpoint is infinite.

Neutrino Oscillations

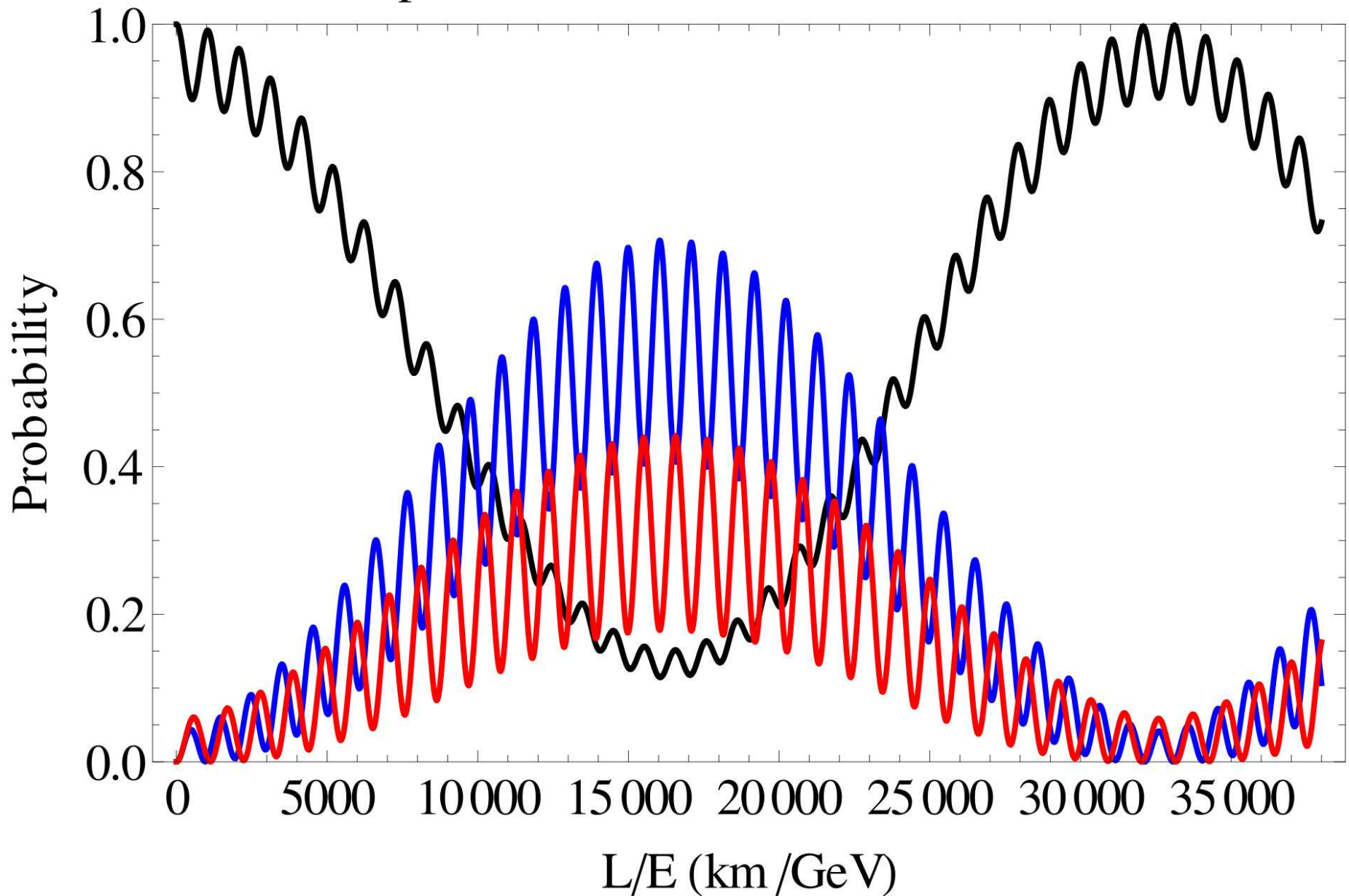
As strange as it may seem, the neutrino is actually a composite of its 3 “flavor” types, e , μ , and τ .

Consequently, the fast, lighter components separate from their heavier counterparts.

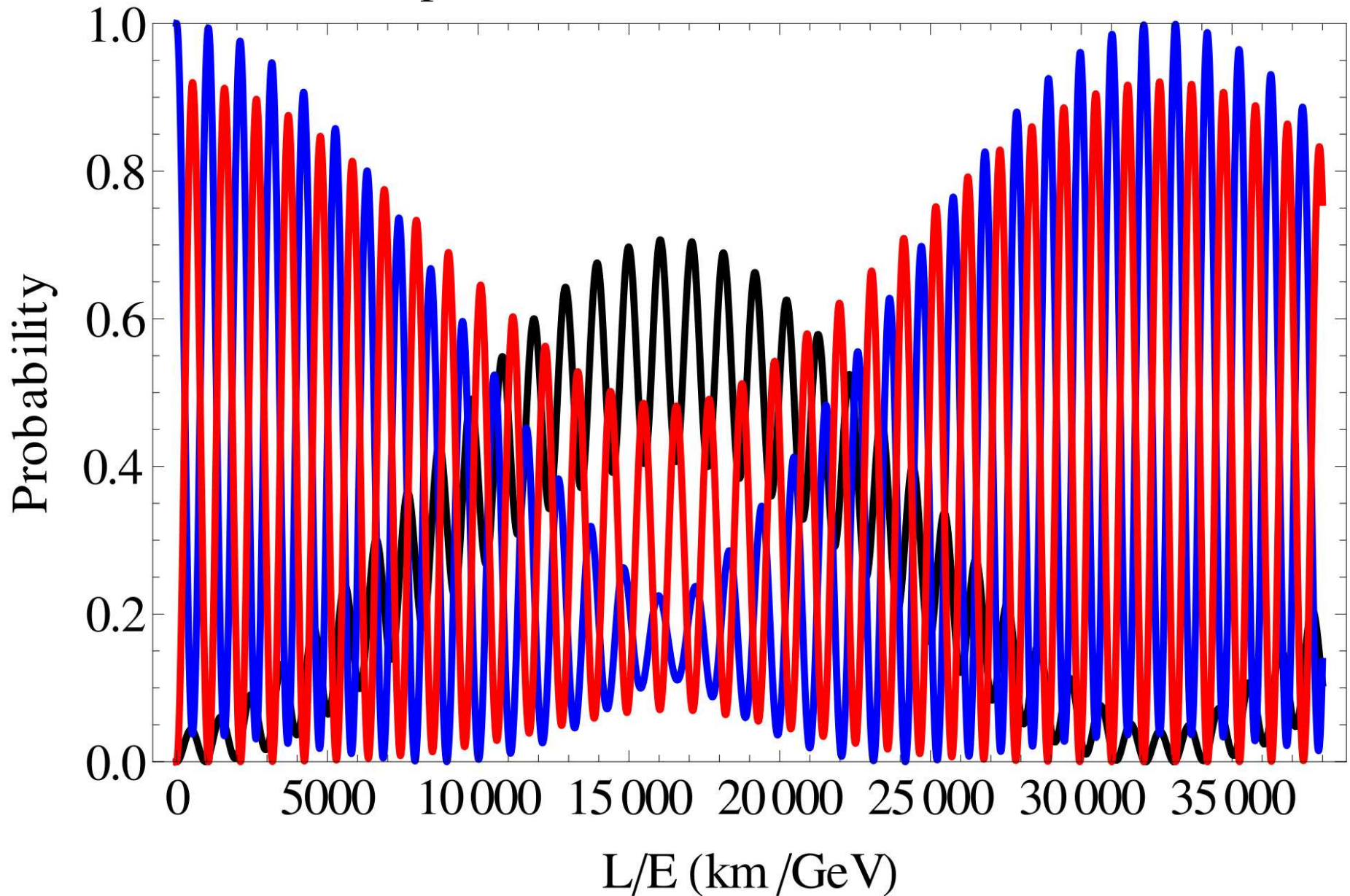
The math is not prohibitive, but lengthy.

However, the figures show the effect quite clearly.

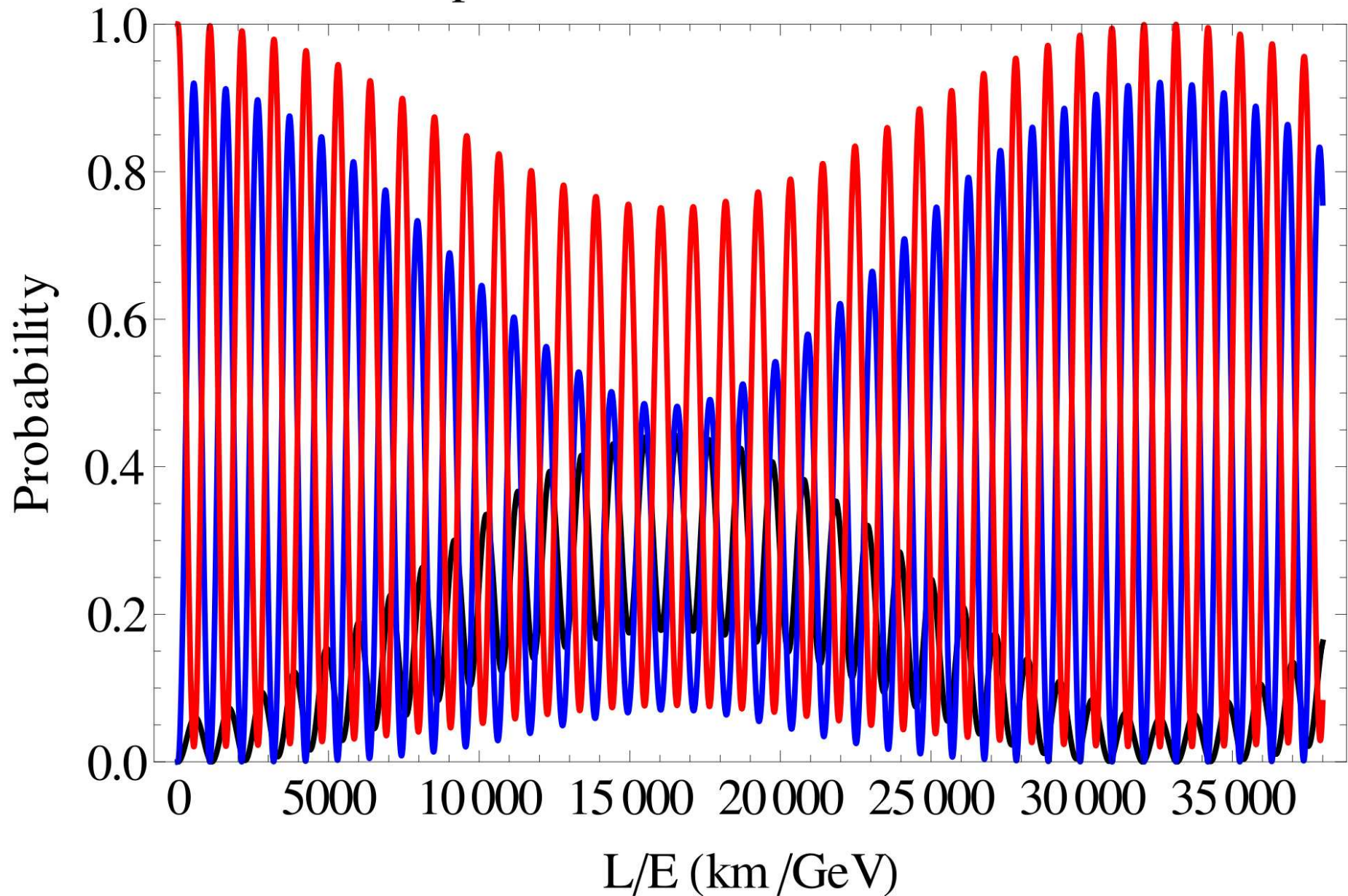
Oscillation probabilities for an initial electron neutrino



Oscillation probabilities for an initial muon neutrino



Oscillation probabilities for an initial tau neutrino



Chapter 15.4: Angular momentum and parity selection rules

Classification of transitions in β decay

The e and the ν in the final states of a β decay each have intrinsic spin- $\frac{1}{2}$. Conservation of total angular momentum requires that:

$$\vec{I}_X = \vec{I}_{X'} + \vec{L} + \vec{S}, \quad (39)$$

where \vec{I}_X , $\vec{I}_{X'}$ are the total angular momenta of the parent and daughter, respectively, and \vec{L} , \vec{S} are the total orbital and total spin angular momentum, respectively, of the $e\nu$ pair.

Therefore, the ΔI can be $\pm L$, for $S = 0$, or $\pm|L \pm 1|$, for $S = 1$.

If $L = 0$, then $\Delta I = \pm 1$.

There are only two cases for lepton spin alignment.

$S = 0$, when the $e\nu$ intrinsic spins anti-align, is called a Fermi transition.

$S = 1$, when the $e\nu$ intrinsic spins align, is called a Gamow-Teller transition.

Generally, as $L \uparrow$, $\lambda \downarrow$, $t_{1/2} \uparrow$, because there is much less overlap of the $e\nu$ wavefunctions with the nucleus.

The entire characterization scheme is given in the table below. .

Type of Transition	Selection Rules	$L_{e\nu}$	$\Delta\pi?$	ft
superaligned	$\Delta I = 0, \pm 1^*$	0	no	$1 \times 10^3 - 1 \times 10^4$
aligned	$\Delta I = 0, \pm 1$	0	no	$2 \times 10^3 - 10^6$
1 st forbidden	$\Delta I = 0, \pm 1$	1	yes	$10^6 - 10^8$
unique** 1 st forbidden	$\Delta I = \pm 2$	1	yes	$10^8 - 10^9$
2 nd forbidden	$\Delta I = \pm 1^{***}, \pm 2$	2	no	$2 \times 10^{10} - 2 \times 10^{13}$
unique 2 nd forbidden	$\Delta I = \pm 3$	2	no	10^{12}
3 rd forbidden	$\Delta I = \pm 2^{***}, \pm 3$	3	yes	10^{18}
unique 3 rd forbidden	$\Delta I = \pm 4$	3	yes	4×10^{15}
4 th forbidden	$\Delta I = \pm 3^{***}, \pm 4$	4	no	10^{23}
unique 4 th forbidden	$\Delta I = \pm 5$	4	no	10^{19}

Table 2: Classification of transitions in β decay.

Notes: (*) $0^+ \rightarrow 0^+$ can only occur via Fermi decay.

(**) Unique transitions are Gamow-Teller transitions where \vec{L} and \vec{S} are aligned.
The shape factors have very simple forms in this case.

(***) For the $n \geq 2$ forbidden transitions, the $\Delta I = \pm(n - 1)$ transition is often associated with the $n - 2$ forbidden transition, being indistinguishable in the measurements of these processes.

Nomenclature alert!

Nomenclature	Meaning
\vec{L}, L	Total orbital angular momentum of the $e\nu$ pair
\vec{S}, S	Total spin angular momentum of the $e\nu$ pair
Fermi (F) transition	$S = 0$: $e\nu$ intrinsic spins anti-align
Gamow-Teller (GT) transition	$S = 1$: $e\nu$ intrinsic spins align
Superallowed	The nucleon that changed form, did not change its shell-model orbital.
Allowed	$L = 0$ transition. $M_{if}^0 \neq 0$. See (27).
n^{th} forbidden	The $e\nu$ pair carry off n units of orbital angular momentum
Unique	\vec{L} and \vec{S} are aligned.

Examples of allowed β decays

This is straight out of Krane.

$^{14}\text{O}(0^+) \rightarrow ^{14}\text{N}^*(0^+)$ must be a pure Fermi decay since it is $0^+ \rightarrow 0^+$.

Other examples are $^{34}\text{Cl} \rightarrow ^{34}\text{S}$, and $^{10}\text{C} \rightarrow ^{10}\text{B}^*$.

$^6\text{He}(0^+) \rightarrow ^6\text{Li}(1^+)$, a $0^+ \rightarrow 1^+$ transition. This must be a pure Gamow-Teller decay.

Other similar examples are $^{13}\text{B}(\frac{3}{2}^-) \rightarrow ^{13}\text{C}(\frac{1}{2}^-)$, and $^{230}\text{Pa}(2^-) \rightarrow ^{230}\text{Th}^*(3^-)$.

$n(\frac{1}{2}^+) \rightarrow p(\frac{1}{2}^+)$ This is a mixed transition. The F transition preserves the nucleon spin direction, the GT transition flips the nucleon spin. (Show drawing.)

β decay can either be of the F type, the GT type or a mixture of both.

We may generalize the matrix element and coupling constant as follows, for allowed decays:

$$gM^0 = g_{\text{F}}M_{\text{F}}^0 + g_{\text{GT}}M_{\text{GT}}^0 = g_{\text{F}}\langle\psi_{X'}|\mathbf{1}|\psi_X\rangle + g_{\text{GT}}\langle\psi_{X'}|\mathcal{O}_{\uparrow\downarrow}|\psi_X\rangle, \quad (40)$$

where $\mathcal{O}_{\uparrow\downarrow}$ symbolizes an operator that flips the nucleon spin for the GT transition.

The operator for the F transition is simply 1, (*i.e.* unity), and just measures the overlap between the initial and final nuclear states.

The fraction of F transitions is:

$$f_F = \frac{g_F^2 |M_F^0|^2}{g_F^2 |M_F^0|^2 + g_{GT}^2 |M_{GT}^0|^2} = \frac{y^2}{1 + y^2} , \quad (41)$$

where,

$$y \equiv \frac{g_F M_F^0}{g_{GT} M_{GT}^0} . \quad (42)$$

Tables of y values are given in Krane on the next page.

Table 9.3 Ratio of Fermi to Gamow-Teller Matrix Elements

	Decay	$y = g_F M_F / g_{GT} M_{GT}$	%F	%GT
Mirror decays	$n \rightarrow p$	0.467 ± 0.003	18	82
	${}^3\text{H} \rightarrow {}^3\text{He}$	0.479 ± 0.001	19	81
	${}^{13}\text{N} \rightarrow {}^{13}\text{C}$	1.779 ± 0.006	76	24
	${}^{21}\text{Na} \rightarrow {}^{21}\text{Ne}$	1.416 ± 0.012	67	33
	${}^{41}\text{Sc} \rightarrow {}^{41}\text{Ca}$	0.949 ± 0.003	47	53
Nonmirror decays	${}^{24}\text{Na} \rightarrow {}^{24}\text{Mg}$	-0.021 ± 0.007	0.044	99.956
	${}^{41}\text{Ar} \rightarrow {}^{41}\text{K}$	$+0.027 \pm 0.011$	0.073	99.927
	${}^{46}\text{Sc} \rightarrow {}^{46}\text{Ti}$	-0.023 ± 0.005	0.053	99.947
	${}^{52}\text{Mn} \rightarrow {}^{52}\text{Cr}$	-0.144 ± 0.006	2	98
	${}^{65}\text{Ni} \rightarrow {}^{65}\text{Cu}$	-0.002 ± 0.019	< 0.04	> 99.96

Chapter 15.4.1: Matrix elements for certain special cases

This section is meant to explain several things given without explanation in Krane's Chapter 9.

$M_{if} = \sqrt{2}$, for superallowed $0^+ \rightarrow 0^+$ transitions

This was stated near the top of the text on Krane's p. 284. ($M_{if} = \sqrt{2}$.)

We know that a $0^+ \rightarrow 0^+$ allowed transition (super or regular), must be an F transition. In the case that it is also a superallowed transition, we can write explicitly:

$$M_{if} = \left\langle \psi_{X'}(0^+) \left(\frac{1}{\sqrt{2}} [e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)] \right) \middle| \mathbf{1} \middle| \psi_X(0^+) \right\rangle, \quad (43)$$

where the intrinsic spins of the $e\nu$ pair are shown explicitly. This spin wavefunction is properly normalized with the $\sqrt{2}$ as shown.

Separating the spins part, and the space part,

$$M_{if} = \frac{1}{\sqrt{2}} \langle \psi_{X'} | \psi_X \rangle \langle (e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)) | \vec{0} \rangle = \sqrt{2}, \quad (44)$$

since $\langle \psi_{X'} | \psi_X \rangle = 1$ for superallowed transitions, and $\langle e(\uparrow)\nu(\downarrow) | \vec{0} \rangle = \langle e(\downarrow)\nu(\uparrow) | \vec{0} \rangle = 1$. Using this knowledge, one can measure directly, g_F from $0^+ \rightarrow 0^+$ superallowed transitions. Adapting (38) for superallowed transitions,

$$g_F^2 = \frac{\log_e(2)\pi^3\hbar^7}{m_e^5 c^4} \left(\frac{1}{ft_{1/2}} \right)_{\text{meas}}, \quad (45)$$

giving a direct measurement of g_F via measuring ft .

Table 9.2 in Krane (page 285) shows how remarkably constant ft is for $0^+ \rightarrow 0^+$ super-allowed transitions. This permits us to establish the value for g_F to be:

$$g_F = 0.88 \times 10^{-4} \text{ MeV} \cdot \text{fm}^3 . \quad (46)$$

Table 9.2 ft Values for $0^+ \rightarrow 0^+$ Superaligned Decays

Decay	ft (s)
$^{10}\text{C} \rightarrow ^{10}\text{B}$	3100 ± 31
$^{14}\text{O} \rightarrow ^{14}\text{N}$	3092 ± 4
$^{18}\text{Ne} \rightarrow ^{18}\text{F}$	3084 ± 76
$^{22}\text{Mg} \rightarrow ^{22}\text{Na}$	3014 ± 78
$^{26}\text{Al} \rightarrow ^{26}\text{Mg}$	3081 ± 4
$^{26}\text{Si} \rightarrow ^{26}\text{Al}$	3052 ± 51
$^{30}\text{S} \rightarrow ^{30}\text{P}$	3120 ± 82
$^{34}\text{Cl} \rightarrow ^{34}\text{S}$	3087 ± 9
$^{34}\text{Ar} \rightarrow ^{34}\text{Cl}$	3101 ± 20
$^{38}\text{K} \rightarrow ^{38}\text{Ar}$	3102 ± 8
$^{38}\text{Ca} \rightarrow ^{38}\text{K}$	3145 ± 138
$^{42}\text{Sc} \rightarrow ^{42}\text{Ca}$	3091 ± 7
$^{42}\text{Ti} \rightarrow ^{42}\text{Sc}$	3275 ± 1039
$^{46}\text{V} \rightarrow ^{46}\text{Ti}$	3082 ± 13
$^{46}\text{Cr} \rightarrow ^{46}\text{V}$	2834 ± 657
$^{50}\text{Mn} \rightarrow ^{50}\text{Cr}$	3086 ± 8
$^{54}\text{Co} \rightarrow ^{54}\text{Fe}$	3091 ± 5
$^{62}\text{Ga} \rightarrow ^{62}\text{Zn}$	2549 ± 1280

$$\underline{M_{if} = 1, \text{ for neutron } \beta \text{ decay, } n \rightarrow p + e^- + \tilde{\nu}_e}$$

This was stated near the top of the text on Krane's p. 290.
($M_F = 1$, for neutron β decay.)

In this case, for an F transition:

$$M_{if} = \left\langle \psi_p \left(\frac{1}{\sqrt{2}} [e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)] \right) \left| \mathbf{1} \right| \frac{1}{\sqrt{2}} [\psi_n(\uparrow) + \psi_n(\downarrow)] \right\rangle , \quad (47)$$