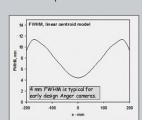


#### V.B.2.d - Resolution

- Variation in the estimate position (x, y) of a detected gamma ray results from statistical noise in the number of electrons collected from each PMT.
- For the centroid estimate shown in slide 18, the estimated X position comes from the observed set of PMT electron signals  $(N_{\rm p}~i=1,8)$  . In the estimate below this is X rather than U(x) .
- The variance of this is computed using propagation of error.
- The spatial resolution in FWHM is then equal to 2.35  $\sigma_{_X}$



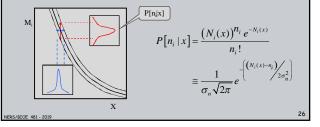
$$X = \sum_{i=1,8} w_i N_i$$
$$(\sigma_x)^2 = \sum_{i=1,8} w_i^2 N_i$$

The FWHM based on the weights used earlier is shown at the left. Poor resolution at the sides results from the high weight values for the outside tubes. This can be avoided by using local estimates with weights adjusted for the approximate position

### V.B.2.d - PMT Position Probability distribution

If the mean number of electrons observed from PMT i for a gamma ray detected at position x is  $N_i(x)$  ,

then the conditional probability of observing  $n_i$  electrons,  $P[n_i|x]$  , is expected to follow a Poisson distribution as was discussed in lecture 05 (or the approximate Gaussian).



#### V.B.2.e - Maximum Likelihood

 If we consider the conditional probabilities associated with each PMT, then the likelihood expression is defined as the product of each.

$$P[n_1, n_2, ..., n_j, | x] = \prod_{i=1}^{j} P[n_i | x]$$

- The maximum likelihood in relation to x can then be taken as as an optimal estimate of the photon interaction position.
- It has been shown that maximizing the log likelihood is equivalent to maximizing the likelihood. This is done by finding the value of x for which the derivative is zero. Using the Poisson distribution, this can be written as.

$$\frac{\delta}{\delta x} \ln \left( P \left[ n_1, n_2, \dots n_j, \mid x \right] \right) = \sum_{i=1}^j n_i \frac{\delta \left( N_i(x) \right) / \delta x}{N_i(x)} - \frac{\delta}{\delta x} \sum_{i=1}^j N_i(x) = 0$$

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# V.B.2.e - Maximum Likelihood

- Clinthorne (IEEE, TNS 1987) shows that the prior equation can be rearranged as a sum of terms linear in  $n_{\rm i}$  ,

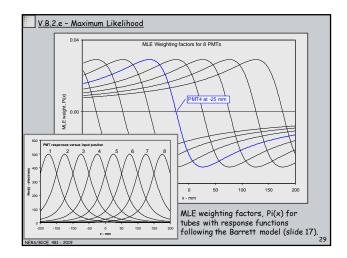
$$0 = \sum_{i=1}^{N} n_i \left[ \frac{\delta \left( N_i(x) \right) / \delta x}{N_i(x)} - e(x) \right] = \sum_{i=1}^{N} n_i p_i(x) \quad , \quad e(x) = \frac{\sum_{j=1}^{i} \delta \left( N_j(x) \right) / \delta x}{\sum_{j=1}^{i} N_k(x)}$$

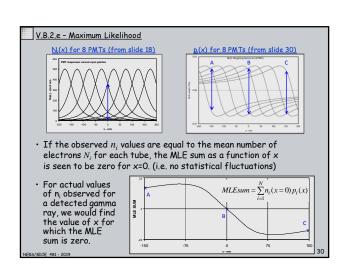
- The second term of the weighting factor, e(x) , compensates for changes in total light collection and is small except at the edges.
- The weighting factor,  $p_i(x)$  , is a function of x approximately equal to the derivative of the PMT response function (see Barrett Fig 5.49).
- The value of x for which this weight sum is zero is the maximum likelihood estimate of the position of the detected gamma ray.
- The variance, which dictates the resolution is otherwise shown to be equivalent to equation 5.205 in Barrett.

$$\left(\sigma(x)\right)^{2} = \left[\sum_{i=1}^{j} \frac{\left(\delta\left(N_{i}(x)\right)/\delta x\right)^{2}}{N_{i}(x)}\right]^{-1}$$

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## V.B.2.e - Maximum Likelihood

As an illustration, consider a system with 7 PMTs at 50 mm intervals. Each have Gaussian response functions with a FWHM of 50 mm:

of 50 mm:  

$$N_{i}(x) = N_{\text{max}}e^{-\left(\frac{x-50i}{W}\right)^{2}}$$
  $W = \frac{50}{2\sqrt{.693}}$ 

- From pg. 28, ignoring the e(x) term we seek a solution to:

$$0 = \sum_{i=-3}^{i=3} \left( \frac{\delta N_i(x)}{\delta x} / N_i(x) \right) n_i$$

· For the Gaussian shape function, the derivative is:

$$\frac{\delta N_i(x)}{\delta x} = -N_i(x)2\bigg(\frac{x-50i}{W^2}\bigg)$$
 · And the MLE equation becomes

$$0 = \sum_{i=-3}^{3} \left\{ -2n_i \left( \frac{x - 50i}{W^2} \right) \right\}$$

$$\begin{split} 0 &= -\sum_{i=-3}^{3} \left\{ 2n_{i} \frac{x}{W^{2}} \right\} + \sum_{i=-3}^{3} \left\{ 2n_{i} \frac{50i}{W^{2}} \right\} \\ \frac{2}{W^{2}} x \sum_{i=-3}^{3} n_{i} &= \frac{2}{W^{2}} 50 \sum_{i=-3}^{3} n_{i} i \\ x &= 50 \sum_{i=-3}^{3} n_{i} \\ \sum_{i=-3}^{3} n_{i} \end{split}$$

• Interestingly, for Gaussian response functions, the MLE solution reduces to a simple centroid the same as the traditional Anger method.

For a set of  $n_i$  values equal to [ 2, 3, 15, 183, 272, 20, 5 ]

we get: 
$$x = 50 \frac{300}{500} = 30 \, mm$$

