X-ray Differential Energy Spectrum

An integral solution of the continuous spectrum based on the radiative cross section in the book by Evans

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Consider a target material and vector of xray energys defined as:

Variables:

$T_o := 100$ Electron energy, keV

$\theta := 10$ x-ray takeoff angle from the target in degrees

Atomic data for tungsten:

$Z := 74$

$A := 183.85$

$\rho := 19.3$ g/cm^3

$sind(x) := \sin\left(x \cdot \frac{\pi}{180}\right)$ function to compute sin of angle in degrees.

$T_m := 511$ rest mass of the electron, $m_o c^2$, in keV

Differential Radiative Cross Section.

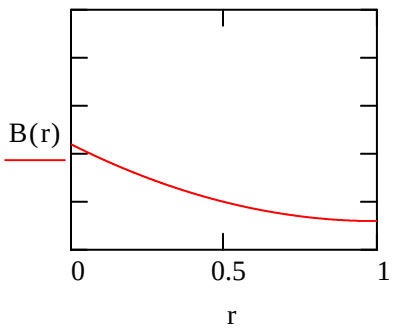
The textbook by Evans describes the principal quantitative results of the quantum-mechanical theory of radiative collisions in the following way. For nuclei of charge Ze, the differential cross section, $\sigma_r(T,E)=d\sigma_r/dE$, for the emission of a photon in the energy range between E and E+dE, by incident electrons of kinetic energy T and total energy $T + m_o c^2$ is given in as:

$\sigma_o := \left(5.80 \cdot 10^{-4}\right) \cdot 10^{-24}$ σ_o is a constant determined by the classical electron radius and the fine structure constant. It is equal to 580 millibarns/nucleus and expressed here with units of cm^2/nucleus

$B(r) := 3.0 + 8.0 \cdot (r-1)^2$ B is a slowly varying function of the relative energy, r=E/T. This function approximates the Heitler result illustrated by Evans for 60 keV. A constant approximation of 4.0 is also defined.

$B_a := 4.0$

$\sigma_r(E,T) := \sigma_o \cdot B\left(\frac{E}{T}\right) \cdot Z^2 \frac{(T + T_m)}{T} \cdot \frac{1}{E}$ cm^2/nuclei/keV



This cross section is understood to be the probability per nuclei of producing a bremsstralung x-ray in the energy range from E to E+dE. We are interested in obtaining the probability per differential pathlength, ds, of the electron as it travels in the medium. This can be obtained by the product of the differential cross section and the number of nuclei per unit volume.

$$N_o := 6.022 \cdot 10^{23}$$ Avogadro's number

$$N := N_o \cdot \frac{\rho}{A}$$ nuclei/cm^3

$$\sigma_{rs}(E,T) := N\,\sigma_r(E,T)$$ probability per cm per keV, cm^{-1}kev^{-1}

This propability is written with double subscripts to remind us that it is doubly differential That is it is the probability of producing a x-ray in the energy range from E to E+dE for a electron of energy T traveling a distance ds, $\sigma_{rs}(T,E)=d\sigma_r/dEds$.

To compute the x-ray energy spectrum, we prefer the probability of producing x-rays as the electron loses energy from T to T+dT rather than the probability per pathlength. The doubly differential cross section is easily converted if we know dT/ds, $\sigma_{rt}(T,E)=(d\sigma_r/dEds)/(dT/ds)$

Electron energy loss per differential pathlength, dt/ds.

For an electron traveling in a material, it is common to assume that it continuously loses energy along it's path. This is referred to as the continuous slowing down approximation, csda. The differential energy loss is referred to as the 'linear energy transfer, LET' or 'stopping power'. We will use the symbol lamda to refer to the LET, $\lambda(T)=dT/ds$. A very good approximation is;

$$k_{sp} := 2.417 \cdot 10^{5}$$ (for T in keV)

$$\lambda(T,Z) := k_{sp}\left(\frac{Z^{.78}}{A}\right)\left(\frac{\rho}{T^{.64}}\right)$$ keV/cm

$$\frac{\lambda(80,74)}{\rho} = 2.285\times 10^{3}$$ as compared with 2.289 for an exact expression.

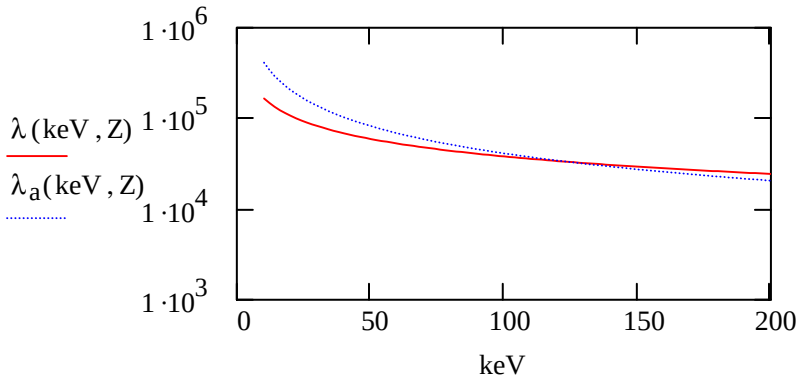
A further approximation that we will consider below assumes that the stopping power is proportional to (Z/A) and inversely proportional to electron energy, (1/T).

$$k_a := 2.2 \cdot k_{sp}$$

$$\lambda_a(T,Z) := k_a\left(\frac{Z}{A}\right)\left(\frac{\rho}{T}\right)$$ keV/cm

$keV := 10, 12 .. 200$

A log-log plot of two approximate expressions for the stopping power in relation to energy



Integral expression for the x-ray spectrum.

We now want to consider an electron of energy To that enters a target and continuously loses energy until it comes to rest. We represent the length of the path traveled when the electron energy has fallen to T to be s(T). The electron will be capable of producing x-rays in the energy range E to E+dE as long as the electron energy is greater than E, T>E. The x-ray differential energy spectrum can thus be written as

$$N(E) = \int_0^{s(E)} \sigma_{rs}(E,T)\,ds = \int_E^{T_o} \frac{\sigma_{rt(E,T)}}{dT/ds}\,dT \qquad \text{x-rays/electron/keV}$$

N(E) is the number of x-rays per electron with energy E to E+dE created while the electron travels a length ds. We use the stopping power to convert this integral to an integration over the electron energy as it is reduced from To to E. Because the stopping power is negative, although conventionally expressed as a positive number, the limits of integration are reversed.

X-ray Spectrum - approximate stopping power and T << Tm.

If we now substitute the expressions above for $\sigma_{rt}(E,T)$ using constand B and the simple approximation for dE/ds given by λ_a , this integral can be written as;

$$N(E) = \int_E^{T_o} \frac{\left(N_o \frac{\rho}{A} \sigma_o B_a Z^2 \left(\frac{T+T_m}{T} \right) \frac{1}{E} \right)}{\left(k_a \frac{Z}{A} \frac{\rho}{T} \right)} dT$$

$$= \left(\frac{N_o \sigma_o B_a}{k_a} \right) Z \frac{1}{E} \int_E^{T_o} (T + T_m)\,dT$$

If we assume that T is small relative to Tm, this is trivially solved to yield;

$$N_a(E) := \frac{\left(N_o \cdot \sigma_o \cdot B_a\right) \cdot T_m}{k_a} \cdot Z \cdot \frac{\left(T_o - E\right)}{E} \text{ ys/electron/keV}$$

This can then be converted to x-rays/steradian by dividing by 4π and to x-rays/mA-s usi the conversion for electrons/mA-s;

$$k_e := 6.24 \cdot 10^{15} \quad \text{electrons/mA-s}$$

$$\phi_a(E) := \left[\frac{k_e}{k_a} \cdot \frac{\left(N_o \cdot \sigma_o \cdot B_a\right) \cdot T_m}{4 \cdot \pi} \right] \cdot Z \cdot \frac{\left(T_o - E\right)}{E} \quad \text{x-rays/mA-s/keV/sr}$$

$$\left[\frac{k_e}{k_a} \cdot \frac{\left(N_o \cdot \sigma_o \cdot B_a\right) \cdot T_m}{4\pi} \right] = 6.667 \times 10^8$$

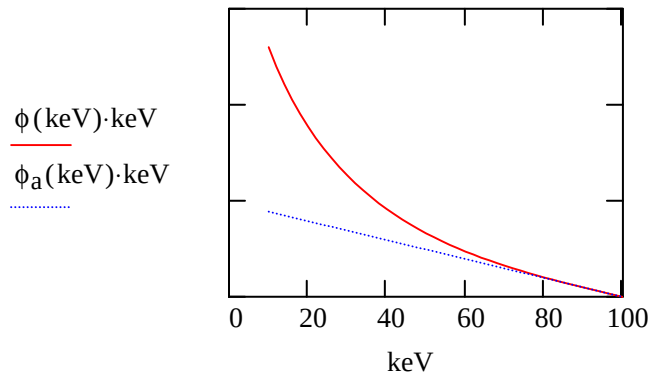
This result is seen to have the same functional relationship with E and Z as for the Kramers relationship with a constant that is slightly larger, 6.7x10^8 as opposed to about 5 to 6.3 x10^8 for the Kramers equation for To = 40 to 180 keV.

X-ray spectrum - numeric solution with improved stopping power.

In general, integral solutions for the x-ray spectrum are done using numerical methods and accurate representations for the cross section and stopping power. The approximation for stopping power defined above, I, is good for Z between about 40 and 80 and for energies between about 5 and 150 keV. For the Evans cross section and this stopping power, mathcad can implements the numeric integration;

$$\phi(E) := \frac{k_e}{4 \cdot \pi} \cdot \int_E^{T_o} \frac{\sigma_{rs}(E, T)}{\lambda(T, Z)} \, dT$$

The spectrum is best plotted as the energy fluence, $\phi(E)E$, as shown to the right. This numeric solution is seen to have significantly more intensity at low energy than the result similar to Kramer's equation. This is due to both the effects of the stopping power and the increased B value at lower x-ray energies.



The effect of self absorbtion in the target.

We now consider the absorbtion of x-rays produced some depth within the target. The model we use involves the electron entering at the surface and penetrating to the mean depth. We use the expression $R_m(T,To)$ to describe the mean depth at a function of the declining electron energy. We simply include an attenuation term inside the integral expression for the differential enery spectrum. Because the x-rays of typical interest are emitted at a small angle relative to the target surface, we must correct the distance over which absorbtion for this obliquity. Any angles for typical line focus x-ray tubes are in the range of 8 to 12 degrees for general radiography tubes and somewhat larger for mammography tubes.

$$F_a(E,T) := \exp\left(-\mu_{74}(E)\cdot R_m(T,T_o)\cdot \frac{1}{\text{sind}(\theta)}\right)$$

Mean penetration into tungsten for 100 keV electrons is about 1.1 microns (i.e. 1.1X10-4 cm. While small, the high attenuation coeficient of tungsten, μ_{74}, provides significant absorbtion at energies below about 30 keV. When adjusted for obliquity, the absorbtion path is considerably longer. The absorbtion over the oblique path of about 7 micron distance is signigicant because of the large cross section for tungsten.

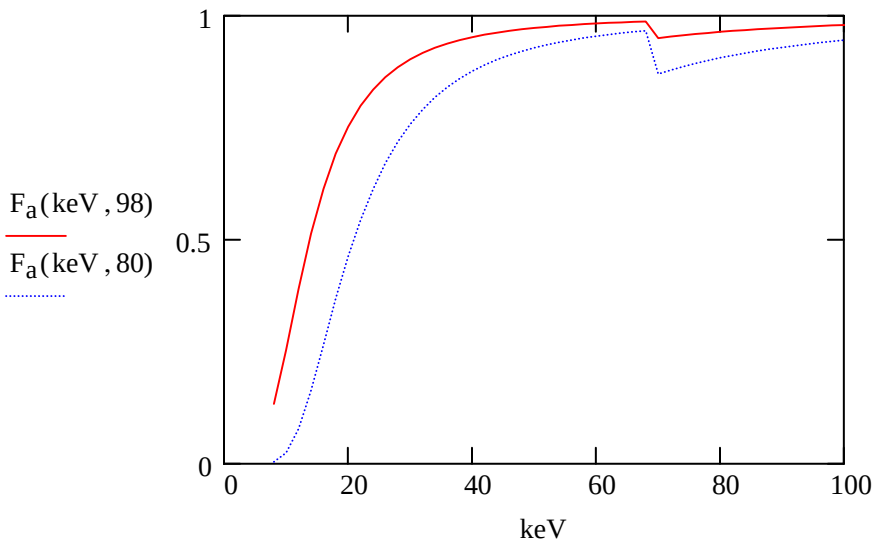
$$R_m(0,100) = 1.151\times 10^{-4} \quad \text{cm}$$

$$\frac{1}{\text{sind}(\theta)} = 5.759$$

$$\mu_{74}(10) = 5.703\times 10^{3} \quad 1/\text{cm}$$

In the graph below, the absorbtion from the tungsten tube being considered is illustrated for x-ray production by electrons that have slowed down to 99 keV and to 80 keV. Below this energy the absorbtion is not significantly more.

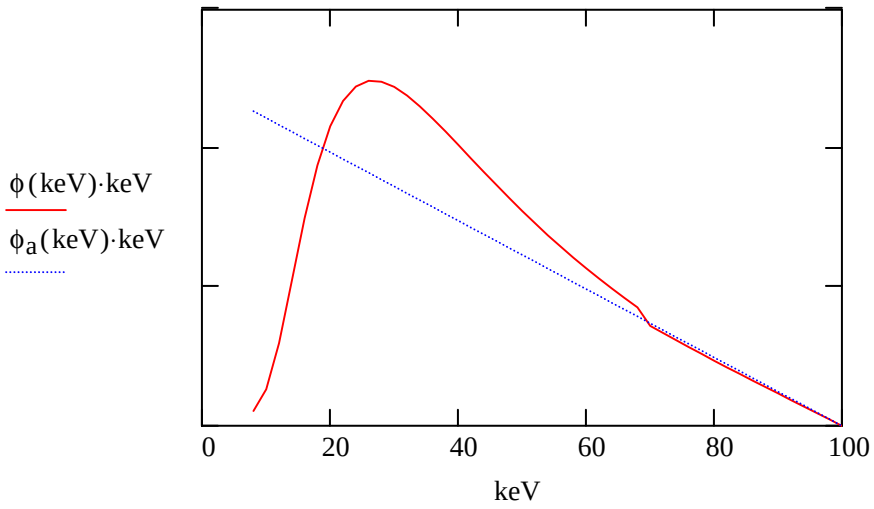
$$keV := 8,10..T_o$$



To compute the differential energy spectrum and account for self absorbtion, we simply need to include the self absorbtion term, $F_a(E,T)$, withing the integral expression that we derived above;

$$\phi(E) := \frac{k_e}{4 \cdot \pi} \cdot \int_E^{T_o} \frac{\sigma_{rs}(E,T)}{\lambda(T,Z)} \cdot F_a(E,T)\, dT$$

Self absorbtion is seen to reduce the excess energy fluence at low energy in relation to the classical Kramer's equation.



The effect of attenuation by the x-ray tube glass and tube housing materials.

Since the cathode and anode of an x-ray tube operate in vacuum, there is always some attenuation of the x-ray spectrum by the tube envelope and additional materials that may be present in the tube housing. The paper by Tucker used above for attenuation coefficients suggest the following values for tube glass (pyrex glass) and the oil and lexan materials in a housing. For completeness, we add the attenuation by 100 cm of air.

$t_{pyr} := .238 \quad t_{oil} := .306 \quad cm$

$t_{lex} := .266 \quad t_{air} := 100$

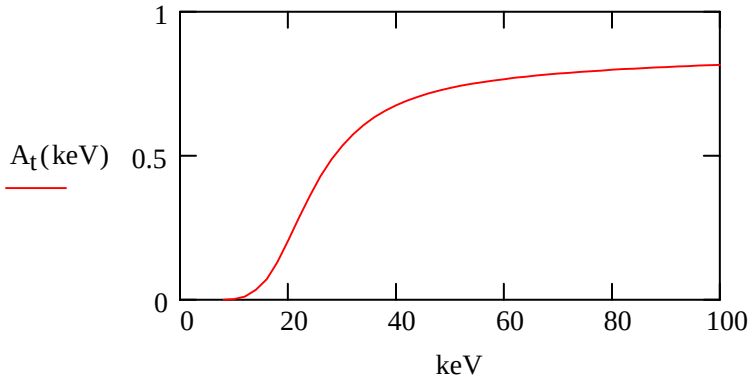
The intrinsic tube/housing/air attenuation is then given by an exponetial expression. Additionally it is common to add some aluminum to the tube housing to further reduce lo energy emission that do not generally reach a radiographic detector but will contribute t radiation dose. We express the effects of added Aluminum filtration in a separate term.

$$A_t(E) := \exp\left[-\left(\mu_{pyr}(E) \cdot t_{pyr} + \mu_{oil}(E) \cdot t_{oil} + \mu_{lex}(E) \cdot t_{lex} + \mu_{air}(E) \cdot t_{air}\right)\right]$$

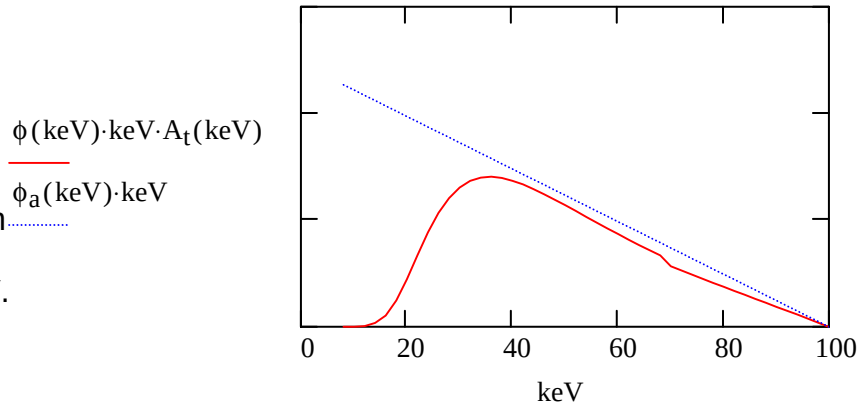
$$A_{al}(E, t_{al}) := \exp\left[-\left(\mu_{al}(E) \cdot t_{al}\right)\right]$$

Note: recall the cross section are not valid below about 15 keV

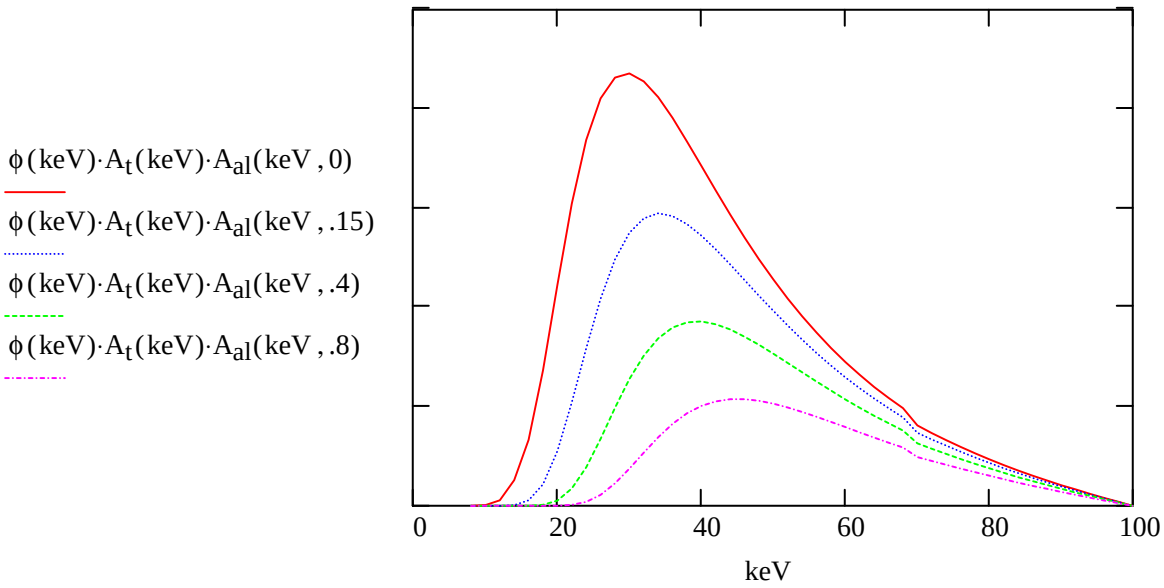
The attenuation by the internal materials of the tube and housing is marked below about 40 keV for general radiography tungsten target tubes. This is commonly referred to as 'intrinsic filtration'.



The effect of intrinsic filtration on the energy fluence spectrum is seen to further reduce low energy emissions such that the spectrum is similar to Kramer's equation above 40keV.



We now include the effects of added filtration and illustrate the shape of the fluence spectrum, photons/sr/mas/keV, for differing amounts of added filtration.



Effect of electron incident energy, kV, and tube takeoff angle.

Two important parameters influencing the shape of the differential x-ray energy spectru are the accelerating voltage that dictates the energy of the incident electron, kV, and the takeoff angle of the tube target. In the mathcad expressions used above, these were defined as constants and set at 100 kV and 10 degrees at the beginning. We can easily include these as variables by redefining the expression for the x-ray spectrum.

$$F_a(E,T,\theta) := \exp\left(-\mu_{74}(E)\cdot R_m(T,T_o)\cdot\frac{1}{\sin(\theta)}\right)$$

For θ in degrees and energies in keV

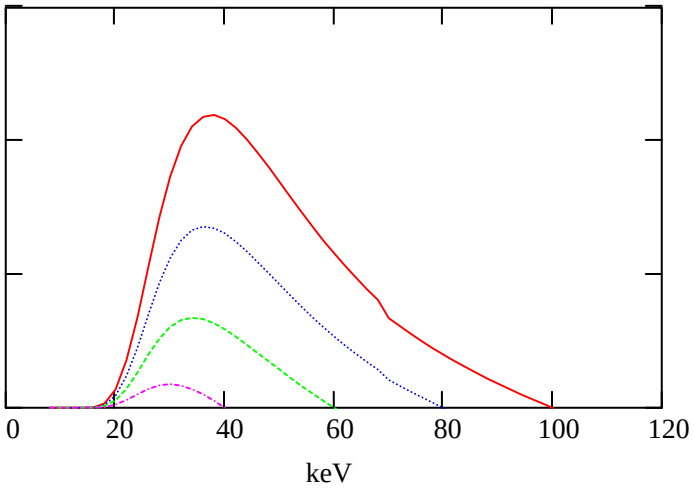
$$\phi(E,T_o,\theta) := \frac{k_e}{4\cdot\pi}\cdot\int_E^{T_o}\frac{\sigma_{rs}(E,T)}{\lambda(T,Z)}\cdot F_a(E,T,\theta)\,dT$$

$\phi(keV,100,10)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,80,10)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,60,10)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,40,10)\cdot A_t(keV)\cdot A_{al}(keV,.3)$



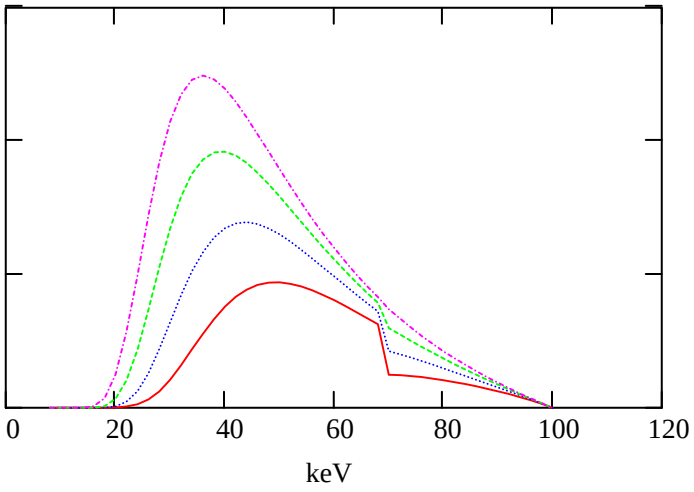
The 'triangular' shape of the spectrum makes the total bremsstralung radiation proportional to $(kV-30)^2$

$\phi(keV,100,1)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,100,2)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,100,5)\cdot A_t(keV)\cdot A_{al}(keV,.3)$

$\phi(keV,100,90)\cdot A_t(keV)\cdot A_{al}(keV,.3)$



The increased self absorbtion with small tube angles is known as the heel effect. At very small angles, the K edge drop is pronounced.