Radiology Research

Henry Ford Health System

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Title:	**<u>Noise Propogation in Computed Tomography</u>**
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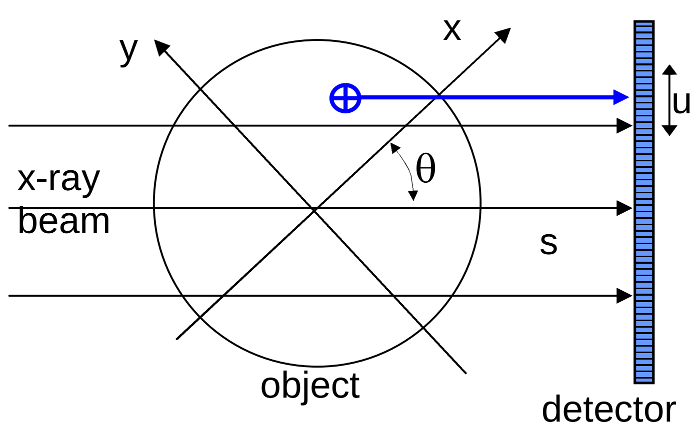
I. Introduction

For computed tomography (CT) systems, measurements of the projection of the linear attenuation coefficient are made at many directions through an object. The figure below shows a parallel beam incident upon a cylindrical object with Cartesian coordinates x and y in the transaxial plane.



The variable u describes the projection position in the detector, θ describes the amount of the rotation of the object about the cylindrical axis, and S describes the path length of the beam through the object. For projections contained in a transaxial plane, projection values as a function of μ and θ are given by the Radon transform[3],

$$P(u,\theta) = \int_0^S \mu(s)ds \, .$$

The integration over ds is intended to imply a path-length integral of the material attenuation, μ(x,y), along the projection. In CT systems, an inverse solution for μ(x,y) from discrete measures of P(u, θ) is performed. This document describes how the noise in measured projection images propagates through the algorithms used to reconstruct the images.

II. Reconstruction Noise Propagation – Parallel Beam.

Reconstruction noise propagation for a parallel beam experiment is derived in this section. Noise propagation for fan beam and cone beam experiments can easily be extrapolated from this geometry. The linear attenuation coefficient for a given location (x, y) in the object is given by the integral over all object rotations of the filtered projection view at location u in the detector.[2]

$$\mu(x,y)=\int_0^{\pi} P^*(u,\theta)d\theta \quad . \tag{1}$$

This can be expressed as a summation for discrete rotations of the object about N_θ equally spaced angles,

$$\mu(x,y)\cong\frac{\pi}{N_\theta}\sum_{k=1}^{N_\theta} P^*(u,\theta_k) \quad . \tag{2}$$

The filtered projection views are obtained by convolving the actual projection view with a kernel,

$$P^*(u,\theta_k)=\int_{-\infty}^{\infty} K(u'-u)P(u',\theta_k)du' \quad . \tag{3}$$

The kernel, K(u'-u), is given by

$$K(u)=\int_{-\infty}^{\infty} F(\omega)|\omega|e^{i\omega u}d\omega \quad , \tag{4}$$

where $|\omega|$ is fundamental to the inverse radon transform and ω is the spatial frequency in the Fourier domain. K(u) is typically an even function for conventional filters. $F(\omega)$ is a filter function that is typically used for noise reduction but can also be used for edge enhancement. If $P^*(u, \theta_k)$ is expressed in discrete form, as

$$P^*(u,\theta_k)=\sum_{l=-\infty}^{\infty} K(l\Delta u-u,\theta_k)P(l\Delta u,\theta_k)\Delta u \quad , \tag{5}$$

then a discrete expression for $\mu(x,y)$ is found by substituting equation (5) into equation (2),

$$\mu(x,y)=\frac{\pi}{N_\theta}\Delta u\sum_{k=1}^{N_\theta}\sum_{l=-\infty}^{\infty} K(l\Delta u-u)P(l\Delta u,\theta_k) \quad . \tag{6}$$

This is in the form of a linear sum of the variables $P(u, \theta)$ multiplied by various constants associated with the kernel values. If the variance of these variables is $\sigma^2_{P(l\Delta u,\theta)}$, we can use error propagation to express the error in $\mu(x,y)$,

$$\sigma^2_{\mu(x,y)}=\left(\frac{\pi}{N_\theta}\right)^2(\Delta u)^2\sum_{k=1}^{N_\theta}\sum_{l=-\infty}^{\infty} K(l\Delta u-u)^2\sigma^2_{P(l\Delta u,\theta_k)} \quad . \tag{7}$$

For x-ray CT systems, the variance in the projections is primarily from statistical noise associated with the limited number of x-rays used in making projection measurements. The calculations presented here do not account for other sources of noise such as detector misalignment, electronic noise, reconstruction artifacts, or insufficient number of projections through the object, or any other noise producing processes that do not arise from the limited number of x-rays used in making the projections.

For cylindrical objects, the projection value, and its associated noise, is equal for projections through the center, i.e. $\sigma^2_{\mu(x,y)} = \sigma^2_{\mu(0,0)} = \sigma^2_{\mu}$, and $\sigma^2_{P(u,\theta)} = \sigma^2_P$. Thus assuming that the noise in each projection is the same,

$$\sigma^2_\mu = \frac{\pi^2}{N_\theta}(\Delta u)^2 \left[\sum_{l=-\infty}^{\infty} K(l\Delta u - u)^2 \right] \sigma^2_P \quad ,$$

or more simply,

$$\sigma^2_\mu = \frac{\pi^2}{N_\theta} \alpha^2 \sigma^2_P \quad , \tag{8}$$

where:

$$\alpha^2 = \Delta u \sum_{l=-\infty}^{\infty} K(l\Delta u - u)^2 \, \Delta u \quad . \tag{9}$$

Following the derivation of Chesler[1], α^2 can be written in integral form

$$\alpha^2 = \Delta u \int_{-\infty}^{\infty} K(u' - u)^2 \, du' \quad . \tag{10}$$

Using Rayleigh’s Theorem, which states that the integral of the squared modulus in the spatial domain is equal to the integral of the squared modulus in the frequency domain*, this can be rewritten as

$$\alpha^2 = \Delta u \int_{-\infty}^{\infty} F^2(\omega) |\omega|^2 \, d\omega \quad . \tag{11}$$

* This theorem is often referred to as Parseval’s Theorem that Bracewell states is a corresponding theorem pertaining to Fourier series instead of continuous functions.

III. Evaluation of α^2 for Specific Filter Functions

When describing the noise propagation, α^2 is a constant that must be evaluated for the specific filter, F(ω), being used. There are a number of different filters used in practice; here two specific filters will be examined.

The first filter evaluated is the ramp function F(ω) = 1 taken at the Nyquist limit, $\omega_{\lim} = 1/2\Delta u$,

$$\begin{aligned}\alpha^2 &= \frac{1}{2\omega_{\lim}} \int_{-\infty}^{\infty} |\omega|^2 d\omega \\ &= \frac{1}{\omega_{\lim}} \int_0^{\omega_{\lim}} \omega^2 d\omega \\ &= \frac{1}{\omega_{\lim}} \left[\frac{\omega^3}{3}\right]_0^{\omega_{\lim}} \\ &= \frac{\omega_{\lim}^2}{3} \ . \end{aligned} \tag{12}$$

Thus, the noise in a reconstruction utilizing a ramp filter is given by

$$\sigma_\mu^2 = \frac{\pi^2}{N_\theta} \frac{\omega_{\lim}^2}{3} \sigma_P^2 \ . \tag{13}$$

Next, the case of a sinc filter, which is often used for noise reduction, is evaluated. In this case,

$$F(\omega) = \frac{\sin\left(\frac{\pi\omega}{\omega_{\lim}}\right)}{\frac{\pi\omega}{\omega_{\lim}}}, 0 < \omega < \omega_{\lim} \ . \tag{14}$$

Since ω is now band limited and F(ω) is symmetric about ω=0, the expression for α^2 in the case of the sinc filter can be rewritten as

$$\alpha^2 = \frac{1}{\omega_{\lim}} \int_0^{\omega_{\lim}} \omega^2 \frac{\sin^2\left(\frac{\pi\omega}{\omega_{\lim}}\right)}{\left(\frac{\pi\omega}{\omega_{\lim}}\right)^2} d\omega$$

$$= \frac{\omega_{\lim}}{\pi^2} \int_0^{\omega_{\lim}} \sin^2\left(\frac{\pi\omega}{\omega_{\lim}}\right) d\omega \ . \tag{15}$$

The integral can be solved by implementing a change of variables such that

$$a = \frac{\pi\omega}{\omega_{\lim}} \ ,$$

and

$$d\omega = \left(\frac{\omega_{\lim}}{\pi}\right) da \ .$$

Thus,

$$\alpha^2 = \left(\frac{\omega_{\lim}}{\pi^2}\right)\left(\frac{\omega_{\lim}}{\pi}\right)\int_0^{\pi} \sin^2 a da$$

$$= \frac{\omega_{\lim}^2}{\pi^3}\int_0^{\pi}\left[\frac{1}{2} - \frac{1}{2}\cos 2a\right] da$$

$$= \frac{\omega_{\lim}^2}{\pi^3}\left[\frac{1}{2}a - \frac{1}{4}\sin 2a\right]_0^{\pi}$$

$$= \frac{\omega_{\lim}^2}{2\pi^2} \ . \tag{16}$$

Thus, the noise in a reconstruction utilizing a sinc filter is given by

$$\sigma_\mu^2 = \frac{\omega_{\lim}^2}{2N_\theta}\sigma_P^2 \ . \tag{17}$$

Since $\omega_{\lim} = 1/2\Delta u$, this can be written in terms of the projection spacing as

$$\sigma_\mu^2 = \frac{\sigma_P^2}{2^3 N_\theta \Delta u^2} \ . \tag{18}$$

The projection spacing refers to the detector sample spacing. This is often the same as the reconstruction pixel size for a parallel beam. When the detector spacing is less than the pixel size, the filter operation for the reconstruction is often limited to a frequency limit given by the pixel size. The spacing for this equation can thus be thought of in terms of the pixel spacing.

IV. Summary

The equations derived in this paper examine how noise due to the fluctuations in the number of photons used in each projection propagates through image reconstruction for images acquired with an x-ray CT system. Specifically, the noise is derived for reconstruction algorithms using the ramp and sinc filters. In general, the noise in a reconstruction is given by equation (8)

$$\sigma_\mu^2 = \frac{\pi^2}{N_\theta}\alpha^2\sigma_P^2 ,$$

where α^2 is given by equation (11) and must be evaluated for the specific filter of interest.

For the ramp filter, where F(ω) = 1, the reconstructed noise is shown in equation (13)

$$\sigma_\mu^2 = \frac{\pi^2}{N_\theta}\frac{\omega_{\lim}^2}{3}\sigma_P^2 \ .$$

For the sinc filter, where

$$F(\omega) = \frac{\sin\left(\frac{\pi\omega}{\omega_{\lim}}\right)}{\frac{\pi\omega}{\omega_{\lim}}}, 0 < \omega < \omega_{\lim} \ .$$

The reconstructed noise is given by equation (17)

$$\sigma_\mu^2 = \frac{\omega_{\lim}^2}{2N_\theta}\sigma_P^2 \ .$$

IV. References

1. Chesler DA, et al: *Noise Due to Photon Counting Statistics in Computed X-Ray Tomography*. Journal of Computer Assisted Tomography, 1(1) 64-74, 1977.

2. Kak, A.C. and Roberts, B.A.: (Chapter 27) *Reconstruction from Projections: Applications in Computer Tomography*. In Young, Tzay Y.: *Handbook of Pattern Recognition and Image Processing*. 649-693; Academic Press, Inc, 1986.

3. Lohner R: Appendix A, *Translation of Radon's 1917 Paper*. School of Mathematics, Georgia Institute of Technology, Atlanta, GA.