

Let $N \rightarrow \infty$, we have $V_\infty(z) \leq J_{\pi_\infty}(z) + \epsilon, \forall \pi_\infty \in \Pi_\infty^s$. Thus, $V_\infty(z) \leq V^*(z) + \epsilon$ and the theorem is proved as ϵ is arbitrary.

Remark 4: Compared with the previous work [3], [11], our convergence result derived specially for the DSLQR problem has several distinctions. First, it allows general terminal cost, which is especially important for the finite-horizon DSLQR problems. In addition, the convergence conditions are expressed in terms of the subsystem matrices rather than the infinite-horizon value function [3], and thus become much easier to verify. Finally, by Theorem 3, for a given tolerance on the optimal cost, the required number of iterations can be predetermined. This provides an efficient means to stop the value iteration with guaranteed suboptimal performance.

V. CONCLUSION

A number of important properties of the value functions of the DSLQR problem are derived. In particular, we have shown that any finite-horizon value function is the pointwise minimum of a finite number of quadratic functions that can be obtained recursively using the SRM. It has also been shown that under some mild conditions, the family of the finite-horizon value functions is homogeneous of degree 2, is uniformly bounded over the unit ball and converges exponentially fast to the corresponding infinite-horizon value function. Future research will focus on employing these properties to efficiently solve the DLQRS problem with guaranteed suboptimal performance.

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Neighboring Extremal Solution for Nonlinear Discrete-Time Optimal Control Problems With State Inequality Constraints

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Abstract—A neighboring extremal control method is proposed for discrete-time optimal control problems subject to a general class of inequality constraints. The approach is applicable to a broad class of systems with input and state constraints, including two special cases where the constraints depend only on states but not inputs and the constraints are over determined. A closed form solution for the neighboring extremal control is provided and a sufficient condition for existence of the neighboring extremal solution is specified.

Index Terms—Model predictive control (MPC), neighboring extremal (NE), optimal control.

I. INTRODUCTION

Solving a nonlinear optimal control problem with a long horizon is generally computationally expensive. In many applications, where a nominal optimal solution is known, it is desired to approximate the optimal control solution when the parameters of the optimal control problem are slightly perturbed. Given an optimal control problem and an optimal solution with nominal initial state, the neighboring extremal (NE) method provides a first-order approximation to the optimal solution corresponding to an initial state perturbed from the nominal value. This approach provides a closed-form solution to the approximation problem by calculating the first-order variation of the optimal control and state as a function of the variation of the initial state. The neighboring extremal solution for unconstrained continuous-time systems is presented in [1], [2], while its discrete-time counterpart can be found in [3] and [4]. Subsequently, the NE solution for continuous-time systems with inequality constraints and discontinuities has been derived using multipoint boundary value techniques, as presented in [5].

For discrete-time systems subject to constraints, the finite horizon optimal control problem, in general, can be reformulated as a nonlinear programming problem. Consequently, exploiting sensitivity analysis for the nonlinear programming problem, the NE solution can be calculated as shown in [6]. If N denotes the length of horizon in the opti-

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mization problem, the computational complexity of the corresponding nonlinear programming problem and of the method in [6] is of the order N^3 .

In an effort to alleviate the computational burden, we developed a NE method for constrained discrete-time systems. These results were presented in a conference paper [7]. However, the results in [7] are not applicable to cases when there are pure state inequality constraints, or when the state inequality constraints are over-determined at some time instants.

A modified NE method is proposed in this technical note for discrete-time systems subject to general inequality constraints on inputs and states. This includes the case when constraints depend only on states but not inputs and the case when the active inequality constraints out-number control inputs at some time instants. These are the two cases which cannot be treated by the results in [7]. To derive the neighboring extremal solution we construct a quadratic programming (QP) problem with equality constraints as an intermediate step. The solution to an analogous QP problem for discrete-time systems subject to constraints has been proposed in the context of efficient sequential quadratic programming (SQP) [8]. Our approach exploits a fast method of first-order approximation of optimal control sequence for a perturbed initial state $x(0)$, provided that the perturbation is sufficiently small. In this technical note, not only is a recursive method for solving the associated QP problem is proposed with the computational complexity of order N as opposed to N^3 , but a sufficient condition for convexity of the QP problem is provided to guarantee the optimality of the solution. To the best of our knowledge, similar sufficient conditions have not been reported for the solution to the QP problem associated with the SQP method. Moreover, to formulate the QP associated with the NE method, an algorithm for computing the Lagrange multipliers corresponding to the constraints and co-states is developed.

We emphasize that our method follows a different approach than [9]–[11], where a similar problem of calculating the perturbed optimal solution is considered in a different context of computationally efficient nonlinear model predictive control. Our approach uses indirect (variational) approach, similar to the method proposed in [12] for unconstrained systems, to calculate the first-order correction to the optimal solution when the initial state is perturbed. However, the other methods of [9]–[11] use the collocation approach or direct multiple shooting method [13]. Moreover, we obtain a second-order sufficient condition of optimality that can be easily checked and guarantees existence of optimal solution for initial states in a neighborhood of the nominal initial state. Furthermore, it guarantees existence of the NE solution. The closed-form of the NE solution presented in this technical note also enables fast computation of the perturbed solution in the case when the activeness of the constraints are intact under initial state perturbation. For a large initial state perturbations that can result in activity status change of the constraints, an approach based on the repeated application of our method can be used—see [7].

The NE approach can be employed in any circumstances where an optimal solution is available a priori and the optimal solution corresponding to new initial state needs to be approximated to avoid the burden of solving again the optimization problem, see [5], [14]. In particular, our results can be used for the development of fast model predictive control (MPC) algorithms—see [15].

II. PROBLEM FORMULATION

Consider the following optimal control problem:

$$\mathcal{P}(x_0) : \min_{u: [0, N] \rightarrow \mathbb{R}^m, x: [0, N] \rightarrow \mathbb{R}^n} J[u(\cdot), x(\cdot)] \quad (1)$$

where

$$J[u(\cdot), x(\cdot)] = \sum_{k=0}^{N-1} L(x(k), u(k)) + \Phi(x(N)) \quad (2)$$

subject to

$$x(k+1) = f(x(k), u(k)), \quad f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \quad (3)$$

$$x(0) = x_0, \quad x_0 \in \mathbb{R}^n \quad (4)$$

$$C(x(k), u(k)) \leq 0, \quad C: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^l \quad (5)$$

$$\bar{C}(x(k)) \leq 0 \quad \bar{C}: \mathbb{R}^n \rightarrow \mathbb{R}^{\bar{l}}. \quad (6)$$

We assume that the functions L, f, C, Φ , and \bar{C} are twice continuously differentiable with respect to their arguments. Here, we separate the state-only constraints \bar{C} from the mixed state-input constraints C .

Let $x^o(k), u^o(k), k \in [0, N]$ be the state and control vector sequences corresponding to the optimal solution in the problem of minimizing (2) subject to the constraints (3)–(5) with the initial condition $x(0)$. The solution x^o, u^o is referred to as the nominal solution.

Let $C^a(x(k), u(k))$ be a vector consisting of those elements of the vector $C(x(k), u(k))$ which correspond to active inequality constraints. That is, $C^a(x(k), u(k)) = \text{empty vector}$ if no inequality constraints are active at the time instant k and $C^a(x(k), u(k)) \in \mathbb{R}^{l'}$, if l' (out of l) constraints are active. Similar definitions apply to \bar{C}^a .

Moreover, let $\mu(k)$ and $\bar{\mu}(k)$ be the Lagrange multipliers associated with constraints C^a and \bar{C}^a and $\lambda(k+1)$ be the Lagrange multiplier associated with the equality constraint (3), which is traditionally referred to as the vector of co-states. We can then define the Hamiltonian function as follows:

$$\begin{aligned} H(x(k), u(k), \lambda(k+1), \mu(k)) \\ = L(x(k), u(k)) \\ + \bar{\mu}(k)^T \bar{C}^a(x(k)) \\ + \lambda(k+1)^T f(x(k), u(k)) \\ + \mu(k)^T C^a(x(k), u(k)) \end{aligned} \quad (7)$$

where C^a and \bar{C}^a are the vectors containing active constraints of C and \bar{C} , respectively.

Before proceeding, we define the following compact notations for partial derivatives that will be used for the rest of the technical note, and where for notational simplicity the dependency of the partial derivatives on x, u has been dropped and replaced by k

$$\begin{aligned} H_{uu}(k) &:= \frac{\partial^2 H}{\partial u^2}(x(k), u(k), \lambda(k+1), \mu(k)) \\ H_{ux}(k) &:= \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u}(x(k), u(k), \lambda(k+1), \mu(k)) \right) \\ H_{xu}(k) &:= H_{ux}(k)^T \\ H_{xx}(k) &:= \frac{\partial^2 H}{\partial x^2}(x(k), u(k), \lambda(k+1), \mu(k)) \\ \bar{C}_x^a(k) &:= \frac{\partial \bar{C}^a}{\partial x}(x(k)) \\ f_u(k) &:= \frac{\partial f}{\partial u}(x(k), u(k)), \quad f_x(k) := \frac{\partial f}{\partial x}(x(k), u(k)) \\ C_u^a(k) &:= \frac{\partial C^a}{\partial u}(x(k), u(k)), \quad C_x^a(k) := \frac{\partial C^a}{\partial x}(x(k), u(k)) \\ &\text{for } k = 0, \dots, N-1 \text{ and} \\ \Phi_x(N) &:= \frac{\partial \Phi}{\partial x}(x(N)). \end{aligned}$$

Since the nominal solution $x^o(\cdot)$ and $u^o(\cdot)$ is optimal, it satisfies the following necessary optimality condition implied by Karush-Kuhn-Tucker (KKT) conditions, namely

$$\begin{aligned} \lambda(k) &= H_x(k), \quad k = 0, \dots, N-1 \\ H_u(k) &= 0, \quad k = 0, \dots, N-1 \\ \lambda(N) &= \frac{\partial \Phi}{\partial x}(x(N)) + \bar{\mu}(N)^T \bar{C}_x(N) \\ \mu(k) &> 0, \quad \bar{\mu}(k) > 0 \quad k = 0, \dots, N. \end{aligned} \quad (8)$$

Definition 2.1: The NE solution refers to the state and control sequences which minimize the second-order variation of the Hamiltonian function $H(\cdot)$ subject to linearized constraints, i.e., it is a solution of the following optimization problem:

$$\begin{aligned} \min_{\delta u(\cdot), \delta x(\cdot)} \quad & \delta^2 \bar{J} \\ \delta^2 \bar{J} &= \frac{1}{2} \delta x(N)^T (\Phi_{xx}(N)) \delta x(N) \\ &+ \frac{\partial}{\partial x} (\bar{C}_x^T(x(N)) \bar{\mu}(N)) \delta x(N) \\ &+ \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} \delta x(k) \\ \delta u(k) \end{bmatrix}^T \\ &\times \begin{bmatrix} H_{xx}(k) & H_{xu}(k) \\ H_{ux}(k) & H_{uu}(k) \end{bmatrix} \begin{bmatrix} \delta x(k) \\ \delta u(k) \end{bmatrix} \end{aligned} \quad (9)$$

subject to the constraints

$$\begin{aligned} \delta x(k+1) &= f_x(k) \delta x(k) + f_u(k) \delta u(k) \end{aligned} \quad (11)$$

$$\delta x(0) = \delta x_0 \quad (12)$$

$$\begin{aligned} C_x^a(x(k), u(k)) \delta x(k) + C_u^a(x(k), u(k)) \delta u(k) \\ = 0 \end{aligned} \quad (13)$$

$$\bar{C}_x^a(x(k)) \delta x(k) = 0. \quad (14)$$

Remark 2.1: It can be verified that the NE solution approximates the optimal state and control sequences for the perturbed initial state, provided that the perturbation is sufficiently small. Specifically, the NE solution is a first-order correction to the optimal state and control sequences so that the necessary conditions for optimality are maintained for the perturbed initial condition.

In the sequel, we develop the NE solution for the problem (1).

III. PERTURBATION ANALYSIS FOR DISCRETE-TIME OPTIMAL CONTROL PROBLEM SUBJECT TO GENERAL CONSTRAINTS

In this section, we derive NE solution for nonlinear systems subject to general input-state constraints. Let us define matrix sequences $\tilde{C}_u(\cdot)$, $\tilde{C}_x(\cdot)$, $\tilde{C}_x^T(\cdot)$, and $S(\cdot)$ using the following backward, in k , recursive equations. Let

$$\begin{aligned} \tilde{C}_x(N) &:= \bar{C}_x^a(x(N)) \\ S(N) &:= \Phi_{xx}(N) + \frac{\partial}{\partial x} (\bar{C}_x^T(x(N)) \bar{\mu}(N)) \end{aligned} \quad (15)$$

and, at the time instant k , we define

$$\begin{aligned} C_{aug}(k) &:= \begin{bmatrix} C_u^a(k) \\ \tilde{C}_x^a(k+1) f_u(k) \end{bmatrix} \\ \tilde{r}_k &:= \text{rank}(C_{aug}(k)). \end{aligned} \quad (16)$$

At each time instant k , there is a matrix $P(k)$ that transforms the matrix $C_{aug}(k)$ into the block form, namely

$$P(k) C_{aug}(k) = [\tilde{C}_u(k)^T \quad 0]^T, \quad (17)$$

with $\tilde{C}_u(k) \in \mathbb{R}^{\tilde{r}_k \times m}$ having independent rows. Note that if $C_{aug}(k)$ is of full row rank, $P(k) = I$, and the zero matrix on the right-hand side of (17) is empty.

By denoting

$$\Gamma(k) := \begin{bmatrix} P(k) \begin{bmatrix} C_x^a(k) \\ \tilde{C}_x^a(k+1) f_x(k) \end{bmatrix} \\ \tilde{C}_x^a(k) \end{bmatrix} \quad (18)$$

and assuming that γ_k is the number of rows of matrix $\Gamma(k)$, we can divide Γ into block matrix as $\Gamma(k) = \begin{bmatrix} \tilde{C}_x(k) \\ \tilde{C}_x(k) \end{bmatrix}$, where \tilde{C}_x , \tilde{C}_x being the first \tilde{r}_k and last $\gamma_k - \tilde{r}_k$ rows of Γ respectively, namely

$$\begin{aligned} \tilde{C}_x(k) &:= [I_{\tilde{r}_k \times \tilde{r}_k} \quad 0_{\tilde{r}_k \times (\gamma_k - \tilde{r}_k)}] \Gamma(k) \in \mathbb{R}^{\tilde{r}_k \times m} \\ \tilde{C}_x(k) &:= [0_{(\gamma_k - \tilde{r}_k) \times \tilde{r}_k} \quad I_{(\gamma_k - \tilde{r}_k) \times (\gamma_k - \tilde{r}_k)}] \Gamma(k) \\ &\in \mathbb{R}^{(\gamma_k - \tilde{r}_k) \times m}. \end{aligned} \quad (19)$$

The above manipulations (16)–(19) represent our constraint back-propagation approach in which the state (11) is used to transform the set of constraints (11)–(14) to the equivalent set of constraints, (11), (12), and $\tilde{C}_x(x(k), u(k)) \delta x(k) + \tilde{C}_u(x(k), u(k)) \delta u(k) = 0$, where $\tilde{C}_u(x(k), u(k))$ is full row rank, as elaborated in (16).

Having $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$ and $Z_{xx}(\cdot)$ defined as

$$\begin{aligned} Z_{uu}(k) &:= H_{uu}(k) + f_u^T(k) S(k+1) f_u(k) \\ Z_{ux}(k) &:= Z_{xu}(k)^T = H_{ux}(k) + f_u^T(k) S(k+1) f_x(k) \\ Z_{xx}(k) &:= H_{xx}(k) + f_x^T(k) S(k+1) f_x(k) \end{aligned} \quad (20)$$

the matrix $S(k)$ for $k < N$ can be defined as follows:

$$\begin{aligned} S(k) &= Z_{xx}(k) - [Z_{xu}(k) \tilde{C}_x^T(k)] K_0(k) \\ &\quad \times [Z_{ux}(k)^T \quad \tilde{C}_x(k)^T]^T \end{aligned} \quad (21)$$

where

$$K_0(k) = \begin{bmatrix} Z_{uu}(k) & \tilde{C}_u(k)^T \\ \tilde{C}_u(k) & 0 \end{bmatrix}^{-1}. \quad (22)$$

Having K_0 defined, we define the matrix K^* as

$$K^*(k) = -[I \quad 0] K_0(k) [Z_{ux}(k)^T \quad \tilde{C}_x(k)^T]^T. \quad (23)$$

Using (15) as an initial condition for backward iteration, we calculate the matrix sequences $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$, $Z_{xx}(\cdot)$, $\tilde{C}_u(\cdot)$, $\tilde{C}_x(\cdot)$, $\tilde{C}_x^T(\cdot)$, $S(\cdot)$ and $P(\cdot)$ according to (17), (19), and (21). The role of matrix manipulation (16)–(19) becomes apparent in (22), where the invertibility of the matrix on the right-hand side requires that $\tilde{C}_u(\cdot)$ is full row rank, once Z_{uu} is strictly positive definite.

Lemma 3.1: If $\tilde{C}_x(0)$ is empty, and

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N-1]$$

then the problem (9) subject to constraints (11)–(14) is convex.

Proof: See Appendix I. ■

It should be noted that the system is linearized at the optimal solution which is time-varying and hence the linearized system is time-varying. The following theorem provides sufficient condition for the existence of the NE solution.

Theorem 3.1: If $\tilde{C}_x(0)$ is empty and $x^o(\cdot)$ and $u^o(\cdot)$ satisfy the necessary condition for optimality (8) and

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N-1] \quad (24)$$

then a NE solution subject to the inequality constraints and initial state perturbation $\delta x(0)$ exists and is unique.

Proof: The condition of $\tilde{C}_x(0)$ being empty implies linear independency of the gradients of active constraints [including equality constraint (3)]. Moreover, the condition (24) guarantees strong convexity of the problem $\mathcal{P}(x_0)$ at the nominal solution $x^o(\cdot)$ and $u^o(\cdot)$.

Therefore, assuming that $x^o(\cdot)$ and $u^o(\cdot)$ satisfy the necessary optimality condition (8) with strict complementarity, i.e., $\mu(k) > 0$ and $\bar{\mu}(k) > 0$, the strong second-order sufficient condition for the problem $\mathcal{P}(x_0)$ is satisfied and therefore the pair $x^o(\cdot)$ and $u^o(\cdot)$ is a strong local minimum. This shows existence of NE solution [17]. Uniqueness is the direct consequence of strong convexity that is implied by (24) and Lemma 3.1. ■

Remark 3.1: If conditions of Theorem 3.1 are satisfied, and Lagrange multipliers associated with inequalities (5) and (6) are strictly positive, then the strong second-order sufficient condition (SSC) for optimality is satisfied at the nominal solution $u^o(\cdot)$ and $x^o(\cdot)$. The SSC has an important consequence that there exists a neighborhood of initial state x_0 , $\mathcal{N}(x_0)$, and continuously differentiable functions $u(k)(x_0) : \mathcal{N}(x_0) \rightarrow \mathbb{R}^m$, for $k = 0, \dots, N-1$, and $x(k)(x_0) : \mathcal{N}(x_0) \rightarrow \mathbb{R}^n$, for $k = 1, \dots, N$, such that for all $x'_0 \in \mathcal{N}(x_0)$:

- 1) the control and state sequences $u(\cdot)(x'_0)$ and $x(\cdot)(x'_0)$ are the optimal solution to the problem (1) with initial state x'_0 ;
- 2) the active constraints corresponding to the optimal solutions $u(\cdot)(x'_0)$ and $x(\cdot)(x'_0)$ are the same as those of the nominal optimal solution $x^o(\cdot) = x(\cdot)(x_0)$ and $u^o(\cdot)(x_0) = u(\cdot)(x_0)$.

These results can be found in [6] and [17]. The NE solution is the first-order approximation of $u(\cdot)(x'_0)$ and $x(\cdot)(x'_0)$ in terms of initial state variation x'_0 in the neighborhood $\mathcal{N}(x_0)$.

Remark 3.2: The convexity condition (24) in the absence of inequality constraints (5)–(6) reduces to the convexity condition provided in [12]. The proposed NE method provides a unified framework to calculate the NE solution and check a sufficient condition for the existence of the solution for systems subject to general constraints, including the unconstrained problem as a special case. However, note that the proof of sufficiency of condition (24) for the existence of NE solution requires a different approach than that in [3], [12]. It should be noted that for continuous-time systems, convexity of the problem in the vicinity of the optimal solution has been considered as a mechanism to assure existence of NE solution [5], [18].

Remark 3.3: It should be noted that only gradients of active constraints are considered when checking the strong SSC in a general nonlinear programming problem.

Remark 3.4: For the infinite length of horizon, i.e., $N = \infty$, the existence of an optimal control sequence $u(\cdot)$ that renders a cost finite is usually guaranteed by stabilizability assumption on the linearized system (3). However, for finite length of horizon, i.e., if $N < \infty$, the cost is always finite and such stabilizability assumption is not required.

Remark 3.5: The condition $\tilde{C}_x(0)$ being empty implies that the constraint back propagation does not produce a constraint on the initial state variation $\delta x(0)$ which is not a variable. The treatment of the case when $\tilde{C}_x(0)$ is empty in the context of receding horizon optimal control is presented in [16].

The following Theorem provides the NE solution for the problem formulated in Section II.

Theorem 3.2: Suppose the perturbation $\delta x(0)$ in the initial state $x(0)$ does not change the activeness status of the constraints, i.e., the optimal solution corresponding to initial state $x(0) + \delta x(0)$ has the same active constraints as the optimal solution to $x(0)$, where $\delta x(0)$ represents a perturbation in initial state. If

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N-1] \quad (25)$$

then the NE solution for the initial state perturbation $\delta x(0)$, i.e., solution to (9), is $\delta x(k)$ and $\delta u(k)$, $k \in [0, N]$ where

$$\delta u(k) = K^*(k)\delta x(k) \quad (26)$$

and K_0 , Z_{ux} , \tilde{C}_x , and K^* are defined in (22), (20), (19), and (23) respectively.

Proof: Due to space limit, sketch of the proof is given in Appendix II. ■

Remark 3.6: The assumption that the constraint activeness status remains unchanged, when the initial state is perturbed, is essential for Theorem 3.2 to hold. Large perturbations for which this assumption may be violated can be handled by repeated application of Theorem 3.2. See [7].

Remark 3.7: Our approach is based on the assumption that an optimal solution is known a priori. If such an a priori known solution is suboptimal, then the NE solution can be modified, as illustrated in [15], to improve it towards optimality. Moreover, if the optimal solution is pre-computed and stored for a grid of initial states, the NE solution can be used to correct the stored optimal solutions and provide a first-order approximation for optimal solutions with initial states inside the grid.

IV. NE ALGORITHM

To implement the NE method, one needs to be able to calculate the Lagrange multipliers $\lambda(\cdot)$, $\mu(\cdot)$ and $\bar{\mu}(\cdot)$. A method for these calculation is proposed in Appendix III.

The procedure for determining the NE solution can be summarized as follows, once the Lagrange multipliers are provided:

- Initialize matrices $S(N)$ and $\tilde{C}_x(N)$ using (15).
- Calculate, in a backward run, matrix sequences $P(\cdot)$ (according to (17)), $\tilde{C}_u(\cdot)$ and $\tilde{C}_x(\cdot)$ (using (17) and (19)), $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$, $Z_{xx}(\cdot)$ (using (20)) and $S(\cdot)$ (using (21)).
- Given initial state variation $\delta x(0)$, in a forward run, calculate $\delta x(\cdot)$ and $\delta u(\cdot)$ using (11) and (26).

V. CONCLUSION

In this technical note we proposed a neighboring extremal control method for discrete-time nonlinear systems with pure state and mixed input-state inequality constraints. The resulting Riccati-like (20)–(22) help reduce the time and effort in computing an approximation to optimal control and state sequences if an initial condition is perturbed. In particular, the computational load grows linearly, in our case, with respect to the length of optimization horizon. Moreover, a SSC for optimality that can be easily checked is provided that guarantees existence of optimal solution for initial states in a neighborhood of the nominal initial state. This condition also guarantees existence and uniqueness of NE solution. The capability of NE solution to provide a fast correction to optimal control solution can benefit several control schemes including fast nonlinear Model Predictive Controllers.

APPENDIX I

PROOF OF LEMMA 3.1

Proof: The existence of NE solution is guaranteed if the quadratic cost (10) is positive definite in the linear variety defined by (11)–(14). Note that, since $\tilde{C}_x(0)$ is an empty matrix, the equality constraints (13) and (14) are equivalent to constraints

$$\tilde{C}_u(k)\delta u(k) + \tilde{C}_x(k)\delta x(k) = 0. \quad (27)$$

It is therefore sufficient to prove that the optimization problem with the cost (9) subject to constraints (11), (12), and (27) is convex.

Suppose at the time instant k , $\text{rank}(\tilde{C}_u(k)) = r_k$, and $\{e_{k,1}, \dots, e_{k,m-r_k}\}$ form a basis for the null space of the matrix $\tilde{C}_u(k)$ such that

$$e_{k,t}^T Z_{uu}(k) e_{k,s} = \begin{cases} 1, & \text{if } t = s \\ 0, & \text{otherwise.} \end{cases}$$

Let us define the vector $\Omega_{k,i}$ as follows

$$\Omega_{k,i} := \begin{bmatrix} \delta x_{k,i}(N)^T, \delta u_{k,i}(N-1)^T, \\ \delta x_{k,i}(N-1)^T, \dots, \delta u_{k,i}(0)^T \end{bmatrix} \quad (28)$$

where $\delta x_{k,i}(\cdot)$ and $\delta u_{k,i}(\cdot)$ are vectors defined as

$$\begin{aligned} \delta u_{k,i}(j) &= 0 \quad j = 0, \dots, k-1 \\ \delta x_{k,i}(j) &= 0 \quad j = 0, \dots, k \\ \delta u_{k,i}(k) &= e_{k,i} \\ \delta x_{k,i}(k+1) &= f_u(k)e_{k,i} \\ \delta x_{k,i}(j+1) &= (f_x(j) + f_u(j)K^*(j))\delta x_{k,i}(j) \\ &\quad j = k+1, \dots, N-1 \\ \delta u_{k,i}(j) &= K^*(j)\delta x_{k,i}(j) \\ &\quad j = k+1, \dots, N-1, \end{aligned} \quad (29)$$

for $k = 0, \dots, N-1$ and $i = 1, \dots, m-r_k$.

In addition, let us define the vector Ω^* as

$$\Omega^* := \begin{bmatrix} \delta x^*(N)^T, \delta u^*(N-1)^T, \delta x^*(N-1)^T, \dots, \delta u^*(0)^T \end{bmatrix} \quad (30)$$

where

$$\begin{aligned} \delta x^*(0) &= \delta x(0), \quad \delta u^*(k) = K^*(k)\delta x^*(k), \text{ and} \\ \delta x^*(k+1) &= (f_x(k) + f_u(k)K^*(k))\delta x^*(k) \\ k &= 0, \dots, N-1. \end{aligned} \quad (31)$$

It can then be shown that $\Omega^* + \left[\left((\Omega_{k,i})_{i=1}^{m-r_k} \right)_{k=1}^{N-1} \right]$ forms exactly a linear variety defined by equations (11)–(14). To show that the problem is convex if $Z_{uu}(k) \succ 0$, $k = 0, \dots, N-1$, it is sufficient to show that

$$\begin{aligned} \delta^2 \bar{J} &:= \frac{1}{2} \delta x_{m,i}(N)^T \Phi_{xx}(N) \delta x_{n,j}(N) \\ &\quad + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} \delta x_{m,i}(k) \\ \delta u_{m,i}(k) \end{bmatrix}^T \begin{bmatrix} H_{xx}(k) & H_{xu}(k) \\ H_{ux}(k) & H_{uu}(k) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \delta x_{n,j}(k) \\ \delta u_{n,j}(k) \end{bmatrix} \\ &= \begin{cases} e_{m,i}^T Z_{uu}(k) e_{m,i}, & \text{if } m = n \text{ and } i = j \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (32)$$

We need the following Lemma in establishing (32):

Lemma 5.1: if $p > m, n$ then

$$\begin{aligned} \delta^2 \bar{J}_p &:= \delta x_{m,i}(p+1)^T S(p+1) \delta x_{n,j}(p+1) \\ &\quad + \begin{bmatrix} \delta x_{m,i}(p) \\ \delta u_{m,i}(p) \end{bmatrix}^T \begin{bmatrix} H_{xx}(p) & H_{xu}(p) \\ H_{ux}(p) & H_{uu}(p) \end{bmatrix} \begin{bmatrix} \delta x_{n,j}(p) \\ \delta u_{n,j}(p) \end{bmatrix} \\ &= \delta x_{m,i}(p)^T S(p) \delta x_{n,j}(p). \end{aligned} \quad (33)$$

Proof: The proof follows by expressing $\delta x_{m,i}(p+1)$ and $\delta x_{n,j}(p+1)$ in terms of $\delta x_{m,i}(p)$ and $\delta x_{n,j}(p)$ from (29). Using definitions (23) and (20), equality (13), and definition of $S(p)$ in (21) we have $\delta^2 \bar{J}_p = \delta x_{m,i}(p)^T S(p) \delta x_{n,j}(p)$.

Continuing with the proof of Lemma 3.1, from (33) and definition of $\delta^2 \bar{J}$ in (32), we see that if $m = n$ then

$$\begin{aligned} \delta^2 \bar{J} &= e_{m,i}^T \left(f_u(p)^T S(p+1) f_u(p) + H_{uu}(p) \right) e_{m,i} \\ &= e_{m,i}^T Z_{uu}(p) e_{m,i}. \end{aligned} \quad (34)$$

If $m \neq n$, without loss of generality, we can assume that $n > m$. Then, it can be shown with the help of Lemma 5.1, that

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_{m,i}(n)^T (f_x(n) + f_u(n)K^*(n))^T \\ &\quad \times S(n+1) f_u(n) e_{n,j} \\ &\quad + \delta u_{m,i}(n)^T H_{uu}(n) e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T H_{xu}(n) e_{n,j}. \end{aligned} \quad (35)$$

Substituting (23) in (35) and using (20), we obtain

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_{m,i}(n)^T Z_{xu}(n) e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T \begin{bmatrix} Z_{ux}(n) \\ \tilde{C}_x(n) \end{bmatrix}^T K_0(n) \\ &\quad \times \begin{bmatrix} -f_u(n)^T S(n+1) f_u(n) \\ 0 \end{bmatrix} e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T \\ &\quad \times \begin{bmatrix} Z_{ux}(n) \\ \tilde{C}_x(n) \end{bmatrix}^T K_0(n) \begin{bmatrix} -H_{uu}(n) \\ 0 \end{bmatrix} e_{n,j} \end{aligned}$$

which can be expressed as

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_{m,i}(n)^T Z_{ux}(n) e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T \begin{bmatrix} Z_{ux}(n) \\ \tilde{C}_x(n) \end{bmatrix}^T K_0(n) \begin{bmatrix} -Z_{uu}(n) \\ 0 \end{bmatrix} e_{n,j}. \end{aligned} \quad (36)$$

Since $e_{n,j}$ belongs to the null space of the matrix $\tilde{C}_u(n)$, the (36) can be written as

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_{m,i}(n)^T Z_{xu}(n) e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T \begin{bmatrix} Z_{ux}(n) \\ \tilde{C}_x(n) \end{bmatrix}^T K_0(n) \begin{bmatrix} -Z_{uu}(n) \\ -\tilde{C}_u(n) \end{bmatrix} e_{n,j}. \end{aligned} \quad (37)$$

Substituting $K_0(n)$ from (22), we have

$$\begin{aligned} \delta^2 \bar{J} &= \delta x_{m,i}(n)^T Z_{xu}(n) e_{n,j} \\ &\quad + \delta x_{m,i}(n)^T \begin{bmatrix} Z_{ux}(n) \\ \tilde{C}_x(n) \end{bmatrix}^T \begin{bmatrix} -I \\ 0 \end{bmatrix} e_{n,j} = 0. \end{aligned} \quad (38)$$

Equation (34) and (38) show that the sufficient condition (32) is satisfied and the proof is complete.

APPENDIX II SKETCH OF THE PROOF OF THEOREM 3.2

Let us assume that $\delta\lambda(\cdot)$, $\delta\mu(\cdot)$ and $\delta\bar{\mu}(\cdot)$ are the Lagrange multipliers associated with constraints (11), (13), and (14), respectively. Hereafter, the superscript a is dropped for notational simplicity, assuming that the constraints appearing in the equations are active.

Applying the KKT conditions to the problem (9)–(14), $\delta x(\cdot)$, $\delta u(\cdot)$, $\delta\lambda(\cdot)$, $\delta\mu(\cdot)$ and $\delta\bar{\mu}(\cdot)$ should satisfy the following equations:

$$\begin{aligned} \delta\lambda(N) &= \left(\Phi_{xx}(N) + \frac{\partial}{\partial x} (\tilde{C}_x^T(x^o(N))\bar{\mu}(N)) \right) \delta x(N) \\ &\quad + \tilde{C}_x^T(x^o(N))\delta\bar{\mu}(N) \end{aligned} \quad (39)$$

$$\tilde{C}_x(x^o(N))\delta x(N) = 0 \quad (40)$$

$$\begin{aligned} \delta\lambda(k) &= H_{xx}\delta x(k) + H_{xu}\delta u(k) + f_x^T(k)\delta\lambda(k+1) \\ &\quad + C_x^T(k)\delta\mu(k) + \tilde{C}_x^T(k)\delta\bar{\mu}(k) \end{aligned} \quad (41)$$

$$\begin{aligned} H_{ux}(k)\delta x(k) + H_{uu}(k)\delta u(k) \\ + f_u(k)^T \delta\lambda(k+1) + C_u(k)^T \delta\mu(k) = 0. \end{aligned} \quad (42)$$

Let us define $\delta\hat{\mu}(N) := \delta\bar{\mu}(N)$ and

$$\begin{aligned}\delta\tilde{\mu}(k) &:= \begin{bmatrix} I_{\tilde{r}_k \times \tilde{r}_k} & 0_{\tilde{r}_k \times (\gamma_k - \tilde{r}_k)} \\ P(k)^{-T} \begin{bmatrix} \delta\mu(k) \\ \delta\hat{\mu}(k+1) \end{bmatrix} \\ \delta\bar{\mu}(k) \end{bmatrix} \\ \delta\hat{\mu}(k) &:= \begin{bmatrix} 0_{(\gamma_k - \tilde{r}_k) \times \tilde{r}_k} & I_{(\gamma_k - \tilde{r}_k) \times (\gamma_k - \tilde{r}_k)} \\ P(k)^{-T} \begin{bmatrix} \delta\mu(k) \\ \delta\hat{\mu}(k+1) \end{bmatrix} \\ \delta\bar{\mu}(k) \end{bmatrix}. \quad (43)\end{aligned}$$

From (15), we have $\delta\lambda(N) = S(N)\delta x(N) + \hat{C}_x^T(N)\delta\hat{\mu}(N)$ and $\hat{C}_x(N)\delta x(N) = 0$. We proceed by induction and assume that at the time instant $k+1$

$$\begin{aligned}\delta\lambda(k+1) &= S(k+1)\delta x(k+1) \\ &\quad + \hat{C}_x^T(k+1)\delta\hat{\mu}(k+1) \quad (44)\end{aligned}$$

$$\hat{C}_x(k+1)\delta x(k+1) = 0 \quad (45)$$

where $\hat{C}_x^T(k)$ and $S(k)$ are calculated from (19) and (21), respectively. Using (16), (17), (11) and (19), (45), (13), and (14) can be written as

$$\tilde{C}_u(k)\delta u(k) + \tilde{C}_x(k)\delta x(k) = 0, \quad \hat{C}_x(k)\delta x(k) = 0 \quad (46)$$

where $\tilde{C}_u(k)$ has independent rows. In addition, using (42), (44), and (17) we have $Z_{uu}(k)\delta u(k) + \tilde{C}_u^T(k)\delta\tilde{\mu}(k) = -Z_{ux}(k)\delta x(k)$, which combined with (46) and subject to the assumption of induction, i.e., (44)–(45), results in (26). Considering definition of $\Gamma(k)$ and substituting $\delta\lambda(k+1)$ in (41), using the expression given by (44) into (41), after algebraic manipulations we obtain

$$\delta\lambda(k) = Z_{xx}(k)\delta x(k) + Z_{xu}(k)\delta u(k) + \tilde{C}_x^T(k)\delta\tilde{\mu}(k) + \hat{C}_x^T(k)\delta\hat{\mu}(k). \quad (47)$$

Applying (26) to (47), we have

$$\delta\lambda(k) = S(k)\delta x(k) + \hat{C}_x^T(k)\delta\hat{\mu}(k). \quad (48)$$

From (48) and (46), we conclude by induction that (26) holds for $k = 0, \dots, N-1$. $\hat{C}_x(0)$ being empty guarantees that (11)–(14) are linearly independent.

APPENDIX III

CALCULATING THE ASSOCIATED LAGRANGE MULTIPLIERS

Assuming the nominal state and control sequences are available, we need to calculate the Lagrange multipliers $\mu(\cdot)$ and $\lambda(\cdot)$ in order to be able to determine the Hessian matrices in (20), so that the neighboring extremal algorithm described above can be applied. In general, numerical algorithms for computing an optimal solution u^0 and x^0 also yield a satisfactory approximation for Lagrange multipliers Λ , μ , and $\bar{\mu}$. An alternative procedure, to calculate Λ , μ , and $\bar{\mu}$ on-line, which avoids to store these values, is given below.

- In a backward run, calculate $D(k)$, with $D(N) = \Phi_x(N)^T$, and $\tilde{\mu}(k)$ using

$$\begin{aligned}\tilde{\mu}(k) &= -(\tilde{C}_u(k)\tilde{C}_u(k)^T)^{-1} \left\{ L_u(k)^T + f_u(k)^T D(k+1) \right\} \\ D(k) &:= L_x(k)^T + f_x(k)^T D(k+1) + \tilde{C}_x(k)^T \tilde{\mu}(k).\end{aligned}$$

- If $\hat{C}_x(0)$ is empty, set $\hat{\mu}(0) = \text{empty matrix}$. Now with $\tilde{\mu}(\cdot)$ and $P(\cdot)$ being avoidable, one can calculate $\mu(k)$, $\hat{\mu}(k+1)$, and $\bar{\mu}(k)$ in a forward run, using¹:

$$\begin{bmatrix} \mu(k)^T & \hat{\mu}(k+1)^T & \bar{\mu}(k) \end{bmatrix} = \begin{bmatrix} P(k)^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\mu}(k)^T & \hat{\mu}(k)^T \end{bmatrix},$$

and $\lambda(k)$ using $\lambda(k) = D(k) + \hat{C}_x(k)^T \hat{\mu}(k)$.

¹Note that number of rows of $P(k)$ is not necessarily equal to that of $\bar{\mu}(k)$.

Remark 5.1: If $\hat{C}_x(0)$ is not empty, there is no equation for calculating $\hat{\mu}(0)$. Therefore, the Lagrange multipliers $\bar{\mu}(0)$, $\mu(0)$ and $\hat{\mu}(1)$ cannot be calculated and the prescribed forward run for computing Lagrange multipliers can not be performed. This case is treated in (16).

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