Automatica 57 (2015) 189–198

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On stability of multiobjective NMPC with objective prioritization[☆]

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ARTICLE INFO

Article history:
Received 28 April 2014
Received in revised form
27 January 2015
Accepted 14 April 2015

Keywords:
Nonlinear model predictive control
Multiobjective control
Stability
Lexicographic optimization
Suboptimality

ABSTRACT

This paper proposes a new multiobjective model predictive control (MO-MPC) of constrained nonlinear systems. According to objective prioritization, the MO-MPC problem is formulated as a lexicographic optimization problem. The optimal solutions are obtained by solving a hierarchy of single objective optimization problems. The conditions guaranteeing the recursive feasibility of the optimization problem and stability of the closed-loop system are derived, which depend only on the most important objective. Moreover, a suboptimal algorithm is presented to reduce the computational demand of MO-MPC. One characteristic of the proposed MO-MPC is that the given objective prioritization is automatically satisfied. The theoretical results are illustrated by a comparison study of an example.

1. Introduction

Multiobjective model predictive control (MO-MPC) has received attention recently, due to its ability to explicitly deal with system constraints and optimize a set of performance criteria systematically and simultaneously over a receding horizon (Maree & Imsland, 2014; Qin & Badgwell, 2003; Rawlings & Mayne, 2009). For most practical control problems, performance criteria often involve multiple conflicting control objectives, such as tracking, economical profit, environmental concerns, etc., which span different levels of relative importance. (See Flores-Tlacuahuac, Morales, & Rivera-Toledo, 2012, Zambrano & Camacho, 2002 and all the references therein.) Unlike the case of single objective MPC (SO-MPC) problems, in general, there is no unique (globally) optimal solution attainable to the MO-MPC problem (Chinchuluun & Pardalos, 2007 and Maree & Imsland, 2014). One feature of interest for the MO-MPC problem is to determine a Pareto optimal solution that satisfies the priorities of the multiple control objectives and guarantees the stability of the MO-MPC controller.

A practical approach for the MO-MPC is to form a scalar cost function being a weighted sum of individual cost functions with the weights that reflect the relative priorities of the multiple objectives. However, selecting a set of appropriate weights is a nontrivial task since reducing a weight on one objective and increasing the other does not necessarily lead to a proportional response in the face of constraints (see, e.g., Long & Gatzke, 2007, Tyler & Morari, 1999 and Vallerio, Van Impe, & Logist, 2014). Furthermore, for such systems as sewer network (Ocampo-Martinez, Ingimundarson, Puig, & Quevedo, 2008), certain objectives are only relevant under specific circumstances. Therefore, the selection of the weights associated with these objectives might not be appropriate when these objectives are irrelevant.

Lately, significant progress in MO-MPC has been reported. For instance, De Vito and Scattolini (2007) optimized linear MPC by minimizing the max of a finite number of objective functions. In Bemporad and Munoz de la Pena (2009) the MO-MPC was designed by minimizing a convex combination of different objective functions and stability of the closed-loop system was guaranteed for the convex combination that is close to the desired convex combination. For nonlinear systems, Magni, Scattolini, and Tanelli (2008) proposed a switched MO-MPC, where the stability was ensured by a state-dependent switch, i.e., the value of the activated cost function must be less than the one of the next activated cost function when the switch occurred. Müller and Allgöwer (2012) exploited the time-dependent switch of multiple cost functions to design MO-MPC of discrete-time nonlinear systems and made use of the average dwell-time method to achieve the stability of the proposed

☆ This work is supported by the National Natural Science Foundation of China (No. 61374111) and Nature Science Foundation of Zhejiang Province (No. LY13F030006). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Riccardo Scattolini under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2015.04.024
0005-1098/

nonlinear MPC (NMPC). In Zavala and Flores-Tlacuahuac (2012), a utopia-tracking MO-NMPC was proposed to minimize the distance of a set of objective functions to its steady-state utopia point, where the stability was guaranteed by the terminal constraint and the assumption of strong duality. Benefits of this scheme are that the controller makes trade-off in the multiple objective functions automatically and the Pareto optimal set does not need to be computed on-line. Moreover, Maree and Imsland (2014) presented a dynamic utopia-tracking MO-NMPC scheme for economic optimization of cyclic processes, in which the recursive feasibility was derived by a cyclic terminal constraint; however, the stability of the resulting closed-loop system is still an open issue.

To handle priorities of multiple objectives effectively, the propositional logic and binary variables were used and therefore the original MO-MPC problem was transformed into a mixed integer nonlinear programming (MINP) (see, e.g., Bemporad & Morari, 1999, Long & Gatzke, 2005 and Vada, Slupphaug, Johansen, & Foss, 2001). In general, the MINP is harder to solve than nonlinear continuous optimization problems. By using the lexicographic optimization, Kerrigan and Maciejowski (2002) presented a general framework for design of MO-MPC with different prioritized objectives, where the MO-MPC problem was formulated by a sequence of single objective MPC problems according to the objective prioritization. Moreover, Ocampo-Martinez et al. (2008) designed a lexicographic MO-MPC for control of sewer network and Padhiyar and Bhartiya (2009) for profile control of distributed parameter systems. Zheng, Wu, Liu, and Ling (2010) proposed a new genetic algorithm to compute the lexicographic MO-NMPC actions. Some merits of the lexicographic MO-NMPC are that it can explicitly take into account the priorities of different objectives to be optimized, no arbitrary weights are used and the Pareto optimal set does not need to be computed at each time. To the best of our knowledge, however, no theoretical results of the feasibility of the lexicographic MO-NMPC problem, the stability and economic optimization have been reported in available literature.

Here we consider a class of MO-MPC problems of constrained nonlinear systems, where the objective functions of interest may be economic costs and conflicting, and are ordered according to their prioritization. The original MO-NMPC problem is then formulated as a lexicographic finite horizon optimal control problem (FHOCP), which is solved via a hierarchy of single objective FHOCPs. Two concepts of feasibility, i.e., hierarchical and horizontal feasibility, are introduced to achieve the recursive feasibility of the lexicographic FHOCP. The conditions for stability are obtained only using the most important objective function. The case of varying objective prioritization is discussed. In order to reduce the computational demand of solving the FHOCP online, a suboptimal prioritized MO-NMPC algorithm is presented. Then a well-known result that "*feasibility implies stability*" (Scokaert, Mayne, & Rawlings, 1999) for single objective MPC is regained. Two key features of the proposed NMPC are that the control actions explicitly rely on the objective prioritization and the stability is dependent only upon the most important objective function. The main contribution of this work is to present the feasibility and stability results of the MO-NMPC scheme subject to objective prioritization. Hence, it is a step forward in stability synthesis of MO-NMPC schemes that explicitly consider various priorities of multiple objectives.

2. Problem setup and preliminaries

Let $\mathbf{I}_{\geq 0}$ denote the set of non-negative integer numbers, $\mathbf{I}_{\geq a}$ be the set $\{i \in \mathbf{I}_{\geq 0} : i \geq a\}$ and $\mathbf{I}_{a:b}$ be the set $\{i \in \mathbf{I}_{\geq 0} : a \leq i \leq b\}$ for some $a \in \mathbf{I}_{\geq 0}$ and $b \in \mathbf{I}_{\geq 0}$. Label '$T$' in superscript denotes the transposition of a vector.

Consider the following discrete-time nonlinear system

$$x_{k+1} = f(x_k, u_k), \quad k \in \mathbf{I}_{\geq 0} \tag{1}$$

where $x_k \in R^n$ and $u_k \in R^m$ are the state and control vectors at sampling time k, respectively, and $f(\cdot,\cdot)$ is a locally Lipschitz function on its arguments with $f(0, 0) = 0$. The system is subject to constraints on the state and control

$$x_k \in X, \qquad u_k \in U, \qquad \forall k \in \mathbf{I}_{\geq 0} \tag{2}$$

where $X \subset R^n$ is a closed set and $U \subset R^m$ is a compact set, both of them containing the origin in their interior. Assume that the states are available for state feedback controllers.

Consider a finite sequence of future control at time k

$$\mathbf{u}_{k,N} = \{u_{0|k}, u_{1|k}, \ldots, u_{N-1|k}\} \tag{3}$$

where the prediction horizon $N \in \mathbf{I}_{\geq 1}$. For a given state x_k and sequence $\mathbf{u}_{k,N}$, the future state of the system at time $k+t$ predicted by using the model (1) at time k is denoted as $x_{t|k}$. Hence, $x_{t+1|k} = f(x_{t|k}, u_{t|k})$ with $x_{0|k} = x_k$. We consider l prioritized objectives of system (1), which are represented by objective cost functions

$$J_i(\mathbf{u}_{k,N}, x_k) = E_i(x_{N|k}) + \sum_{t=0}^{N-1} L_i(x_{t|k}, u_{t|k}), \quad i \in \mathbf{I}_{1:l} \tag{4}$$

where the stage costs $L_i : X \times U \to R$ and the terminal costs $E_i : X \to R$ are continuous on their arguments, $i \in \mathbf{I}_{1:l}$ and $l \in \mathbf{I}_{\geq 2}$. In this paper, the objective functions (4) are assumed to be conflicting and there is no solution optimizing all objectives at the same time.[2] Therefore, additional mechanisms must be used to balance these objectives. Here we exploit the objective prioritization to compute the optimal control sequence.

Without loss of generality, we assume that the objective functions are in the order of importance so that J_1 is the most important and J_l the least important to decision makers. According to this objective prioritization, we define a prioritized multiobjective FHOCP

$$\min_{\mathbf{u}_{k,N}} J(\mathbf{u}_{k,N}, x_k) \tag{5a}$$

$$\text{s.t.} \quad \begin{aligned} & x_{t+1|k} = f(x_{t|k}, u_{t|k}), \quad x_{0|k} = x_k \\ & x_{t+1|k} \in X, \qquad u_{t|k} \in U, \quad t \in \mathbf{I}_{0:N-1} \end{aligned} \tag{5b}$$

where the current state $x_k \in X$, decision vector $\mathbf{u}_{k,N}$ is given by (3) and $J(\mathbf{u}, x)$ is the objective function vector

$$J(\mathbf{u}, x) = [J_1(\mathbf{u}, x), J_2(\mathbf{u}, x), \ldots, J_l(\mathbf{u}, x)]^T \tag{6}$$

which maps the constrained control sequence $\mathbf{u}$ and current state x to a set of values of l objective functions (4). Here the optimization of the vector is defined in the sense of the dominance notion (Marler & Arora, 2004), i.e., an objective function vector $J(\mathbf{u}^*, x)$ is non-dominated if and only if there does not exist another vector $J(\mathbf{u}, x)$ such that $J(\mathbf{u}, x) \leq J(\mathbf{u}^*, x)$ with at least one $J_i(\mathbf{u}, x) < J_i(\mathbf{u}^*, x_k)$.

In SO-NMPC, the optimal control sequence is computed by minimizing a single objective function at each time. In contrast to SO-NMPC, the MO-NMPC must minimize l different (conflicting) objective functions at each time. Therefore, there is typically no single optimal solution but rather a set of possible non-dominant solutions of equivalent quality (Abraham & Jain, 2005). The Pareto optimality is an effective measure of the equivalent quality in multiobjective optimization problems (Chinchuluun & Pardalos, 2007; Ehrgott, 2005). Let $\mathbf{u}^*_{k,N}$ be one of the Pareto optimal

[2] In the general case when some cost functions are consistent, we combine them into a single one.

solutions to problem (5). By the receding horizon principle, the MO-NMPC law is identified as

$$u_k^{mpc} = u_{0|k}^*, \quad k \in \mathbf{I}_{\geq 0}. \tag{7}$$

The goal of the paper is to develop an MO-NMPC scheme that minimizes the objective functions (4) in the order of objective prioritization while stabilizing system (1) at the origin in the face of constraints (2). To this end, we first introduce the following definitions.

Definition 1. A sequence $\mathbf{u}_{k,N}$ is called a feasible solution to problem (5) if it satisfies the constraints in (5) for a given $x_k \in X$.

Definition 2 (*Chinchuluun & Pardalos, 2007*)**.** A feasible solution $\tilde{\mathbf{u}}_{k,N}$ is Pareto optimal for (5) if and only if there exists no other feasible solution $\mathbf{u}_{k,N}$ such that $J_i(\mathbf{u}_{k,N}, x_k) \leq J_i(\tilde{\mathbf{u}}_{k,N}, x_k)$ for all $i \in \mathbf{I}_{1:l}$ and $J_j(\mathbf{u}_{k,N}, x_k) < J_j(\tilde{\mathbf{u}}_{k,N}, x_k)$ for at least one index $j \in \mathbf{I}_{1:l}$.

Definition 3 (*Kerrigan & Maciejowski, 2002*)**.** A feasible solution $\tilde{\mathbf{u}}_{k,N}$ is lexicographic optimal for (5) if and only if there exist no other feasible solution $\mathbf{u}_{k,N}$ and an $i^* \in \mathbf{I}_{2:l}$ such that $J_{i*}(\mathbf{u}_{k,N}, x_k) < J_{i*}(\tilde{\mathbf{u}}_{k,N}, x_k)$ and $J_i(\mathbf{u}_{k,N}, x_k) = J_i(\tilde{\mathbf{u}}_{k,N}, x_k)$ for all $i \in \mathbf{I}_{1:i*-1}$.

In other words, for a lexicographic optimal solution, no objective value can be further reduced without increasing at least one of the higher-prioritized objectives. A standard method for finding a lexicographic solution is to solve a sequentially ordered single objective constrained FHOCPs (Ehrgott, 2005; Marler & Arora, 2004), i.e.,

$$\begin{aligned} J_1^*(x_k) &= \min_{\mathbf{u}_{k,N}} \{J_1(\mathbf{u}_{k,N}, x_k) | (5b)\}, \\ J_i^*(x_k) &= \min_{\mathbf{u}_{k,N}} \left\{ J_i(\mathbf{u}_{k,N}, x_k) \middle| \begin{matrix} J_j(\mathbf{u}_{k,N}, x_k) = J_j^*(x_k), \\ (5b), \quad \forall j \in \mathbf{I}_{1:i-1} \end{matrix} \right\} \end{aligned} \tag{8}$$

for all $i \in \mathbf{I}_{2:l}$ and derive a lexicographic solution

$$\mathbf{u}_{k,N}^* = \arg\min_{\mathbf{u}_{k,N}} \left\{ J_l(\mathbf{u}_{k,N}, x_k) \middle| \begin{matrix} J_j(\mathbf{u}_{k,N}, x_k) = J_j^*(x_k), \\ (5b), \quad \forall j \in \mathbf{I}_{1:l-1} \end{matrix} \right\}. \tag{9}$$

In order to improve numerical computation, the constraints $J_j(\mathbf{u}_{k,N}, x_k) = J_j^*(x_k)$ in (8) and (9) are often relaxed as

$$J_j(\mathbf{u}_{k,N}, x_k) \leq J_j^*(x_k) + \varepsilon_j, \quad \forall j \in \mathbf{I}_{1:l-1} \tag{10}$$

where $\varepsilon_j \geq 0$ are small tolerances determined by decision-makers. For simplicity, hereafter the inequalities are written as $J_j(\mathbf{u}_{k,N}, x_k) \leq J_j^*(x_k)$.

3. Stabilizing prioritized MO-NMPC

We consider the prioritized multiobjective FHOCP in (5) and introduce the following assumptions of the objective function $J_1(\mathbf{u}, x)$.

Assumption 1. The stage cost $L_1(x, u)$ and terminal cost $E_1(x)$ are positive-definite functions with respect to their arguments.

Assumption 2. There exist an invariant set $\Omega \subseteq X$ of system (1), containing the origin in its interior, and a local control law $u = \kappa_1^{loc}(x)$ such that

$$\begin{aligned} &\kappa_1^{loc}(x) \in U, \qquad f(x, \kappa_1^{loc}(x)) \in \Omega, \\ &E_1(f(x, \kappa_1^{loc}(x))) - E_1(x) + L_1(x, \kappa_1^{loc}(x)) \leq 0. \end{aligned} \tag{11}$$

for any $x \in \Omega$.

Remark 1. There are some well-established methods to compute local control law $\kappa_1^{loc}(x)$ and invariant set Ω. If terminal cost $E_1(x)$ is a local Lyapunov function of system (1), its level set can be used as a terminal set and a local control law can be obtained using the methods proposed in e.g., Chen and Allgöwer (1998), Fontes (2001) and Lazar, Heemels, Weiland, and Bemporad (2006). Another approach is to approximate the nonlinear system by a linear differential inclusion and to compute a linear local control law and the maximal invariant set, which is a polyhedron (see, e.g., Chen, O'Reilly, & Balance, 2003 and Yu, Chen, Böhm, & Allgöwer, 2009).

By Assumptions 1 and 2, the problem (5) is reformulated by a lexicographic FHOCP, i.e.

$$J_1^*(x_k) = \min_{\mathbf{u}_{k,N}} \{J_1(\mathbf{u}_{k,N}, x_k) | (5b), x_{N|k} \in \Omega\} \tag{12a}$$

$$J_2^*(x_k) = \min_{\mathbf{u}_{k,N}} \left\{ J_2(\mathbf{u}_{k,N}, x_k) \middle| \begin{matrix} J_1(u_{k,N}, x_k) \leq J_1^*(x_k), \\ (5b), \ x_{N|k} \in \Omega \end{matrix} \right\} \tag{12b}$$

$$\vdots$$

$$J_i^*(x_k) = \min_{\mathbf{u}_{k,N}} \left\{ J_i(\mathbf{u}_{k,N}, x_k) \middle| \begin{matrix} J_j(\mathbf{u}_{k,N}, x_k) \leq J_j^*(x_k), \\ (5b), \ x_{N|k} \in \Omega, \ \forall j \in \mathbf{I}_{1:i-1} \end{matrix} \right\} \tag{12c}$$

for all $i \in \mathbf{I}_{2:l}$, where the terminal region $\Omega \subseteq X$ and $J_i^*(x_k)$ is the optimal value function of the ith-layer FHOCP. That is, the most important objective function is minimized subject to the original constraints. If this subproblem is feasible and has a unique solution, it is the solution to the whole optimization problem. Otherwise, the second most important objective function is minimized by adding a new constraint which guarantees that the most important objective functions preserves its optimal value. If this subproblem is feasible and has a unique solution, it is the solution to the original problem. Otherwise, the process continues. The whole process repeats at each time step. This procedure presents the sequential solution approach to determine the lexicographic optimal solution to the whole problem (12), and is summarized by the following algorithm.

Algorithm 1 (*Lexicographic MO-NMPC Algorithm*)**.**

(1) Input the objective functions $J_i(\mathbf{u}_{k,N}, x_k)$ in (4) and their priorities; set $k = 0$.
(2) Measure the state x_k at time step k.
(3) Compute the optimal control sequence $\mathbf{u}_{k,N}^*$ using the following process:
 (3.1) Solve the first-layer subproblem (12a) and obtain one of optimal sequences, $u_{k,N}^{1*}$;
 (3.2) Solve the ith-layer of problem (12a)–(12c) for all $i \in \mathbf{I}_{2:l}$ and obtain one of optimal sequences, $\mathbf{u}_{k,N}^{i*}$;
 (3.3) Determine a lexicographic optimal sequence of the whole problem (12), $\mathbf{u}_{k,N}^* = \mathbf{u}_{k,N}^{l*}$.
(4) Apply the first element $u_k^{mpc} = u_{0|k}^*$ of the lexicographic optimal sequence to system (1).
(5) Set $k = k + 1$ and go to step (2). ■

In this work, the prioritized MO-NMPC obtained by Algorithm 1 is called the *lexicographic MO-NMPC scheme*. Compare to other multiobjective schemes such as the scalarization scheme (Chinchuluun & Pardalos, 2007) and the utopia-tracking method (Zavala & Flores-Tlacuahuac, 2012, etc.), the lexicographic MO-NMPC explicitly takes into account the priority constraint of the objective functions to be minimized. If the first-layer subproblem is initially feasible, the feasibility of the whole lexicographic problem and the stability of the closed-loop system can be guaranteed.

Note that for the SO-NMPC, the feasibility of the FHOCP implies that the feasibility of the problem at time $k + 1$ is achieved by its feasibility at time k (Rawlings & Mayne, 2009). In other words, a feasible solution to the FHOCP at time $k+1$ can be constructed by some feasible solutions to the problem at time k. However, the feasibility of the lexicographic MO-NMPC at time $k+1$ cannot be achieved by directly using its feasibility at time k since the original, prioritized multiobjective FHOCP is transformed into an l hierarchies of single objective FHOCP. To address this issue, we here introduce two concepts of feasibility of lexicographic FHOCPs.

Definition 4. The lexicographic FHOCP (12a)–(12c) has the hierarchical feasibility at time k if the feasibility of the ith-layer subproblem implies the feasibility of the $(i+1)$th-layer subproblem for all $i \in \mathbf{I}_{1:l-1}$.

Definition 5. The lexicographic FHOCP (12a)–(12c) has the horizontal feasibility if the feasibility of the whole problem at time $k \in \mathbf{I}_{\geq 0}$ implies its feasibility at time $k+1$.

In what follows, we give the results of the feasibility and stability of the lexicographic MO-NMPC.

Theorem 1. *Under Assumptions 1–2, the lexicographic FHOCP (12a)–(12c) admits the hierarchical feasibility at each time $k \in \mathbf{I}_{\geq 0}$.*

Proof. Consider the lexicographic FHOCP (12a)–(12c) with an initial state x_k at time $k \in \mathbf{I}_{\geq 0}$. Assume that the ith-layer subproblem of the whole problem (12) is feasible. Let $\bar{\mathbf{u}}^i_{k,N}$ be an optimal solution to this ith-layer subproblem. Then we have $J_j(\bar{\mathbf{u}}^i_{k,N}, x_k) \leq J^*_j(x_k)$ for all $j \in \mathbf{I}_{1:i-1}$ and terminal state $x_{N|k} \in \Omega$. Substituting $\bar{\mathbf{u}}^i_{k,N}$ into the $(i+1)$th-layer subproblem, it is derived that the constraint (5b) and terminal constraint $x_{N|k} \in \Omega$ are fulfilled. Moreover, we have $J_i(\mathbf{u}_{k,N}, x_k) = J^*_i(x_k)$ when $\mathbf{u}_{k,N} = \bar{\mathbf{u}}^i_{k,N}$ since $\bar{\mathbf{u}}^i_{k,N}$ is an optimal solution to the objective function $J_i(\mathbf{u}_{k,N}, x_k)$. Combining these implies that $\bar{\mathbf{u}}^i_{k,N}$ satisfies all constraints in the $(i+1)$th-layer subproblem. Hence, $\bar{\mathbf{u}}^i_{k,N}$ is a feasible solution to the $(i+1)$th-layer subproblem at time k, i.e., the $(i+1)$th-layer subproblem is feasible at time k.

This completes the proof of Theorem 1. ■

Theorem 2. *Suppose that Assumptions 1–2 hold and the first-layer subproblem (12a) is feasible at time $k \in \mathbf{I}_{\geq 0}$. Then the lexicographic FHOCP (12a)–(12c) admits the horizontal feasibility.*

Proof. Consider the assumption that the first-layer subproblem (12a) is feasible at time $k \in \mathbf{I}_{\geq 0}$. Setting $i = 1$ and by induction, it is known from Theorem 1 that the whole problem (12) is feasible at the same time k.

Let $\mathbf{u}^*_{k,N} = \{u^*_{0|k}, u^*_{1|k} \dots, u^*_{N-1|k}\}$ be an optimal solution to the whole problem (12) at time k. From Algorithm 1, we have $\mathbf{u}^*_{k,N} = \mathbf{u}^{l*}_{k,N}$, where $\mathbf{u}^{l*}_{k,N}$ is an optimal solution to the lth-layer subproblem of the whole problem (12).

In order to find a feasible solution to the problem (12) at time $k+1$, we consider the following control sequence:

$$\tilde{\mathbf{u}}^1_{k+1,N} = \{u^*_{1|k}, \dots, u^*_{N-1|k}, \kappa_1^{loc}(x_{N|k})\} \tag{13}$$

where the control law $\kappa_1^{loc}(x)$ satisfies Assumption 2. Since the state $x_{N|k} \in \Omega$, it is known from Assumption 2 that the control action $\kappa_1^{loc}(x_{N|k}) \in U$ and the terminal state $x_{N|k+1} \in \Omega$. This suggests that the sequence defined by (13) is a feasible solution to the first-layer subproblem (12a) at time $k+1$. By induction and applying Theorem 1 again, we have that the lexicographic FHOCP (12a)–(12c) is feasible at $k+1$, i.e., the whole problem (12) admits the horizontal feasibility. ■

Note that the control sequence (13) is a feasible solution to the first-layer subproblem (12a) at time $k+1$ but not necessarily a feasible solution to the whole problem (12) at $k+1$. From the result of Theorem 2, we define the admissible set $Z(N)$ as this set of $(x, \mathbf{u}_{k,N})$ pairs

$$Z(N) = \left\{ (x, \mathbf{u}_{k,N}) \middle| \begin{array}{l} x_{t+1|k} = f(x_{t|k}, u_{t|k}), \quad x_{0|k} = x, \\ x_{t|k} \in X, \quad u_{t|k} \in U, \quad x_{N|k} \in \Omega, \quad t \in \mathbf{I}_{0:N-1}. \end{array} \right\}.$$

The set of admissible states $X^{mpc}(N)$ is then defined as the projection of $Z(N)$ onto X

$$X^{mpc}(N) = \left\{ x \in X | \exists \mathbf{u}_{k,N} \in U^N \text{ s.t. } (x, \mathbf{u}_{k,N}) \in Z(N) \right\}$$

where U^N is the product of N sets U. Now the stability of the lexicographic NMPC obtained by Algorithm 1 is presented by Theorem 3.

Theorem 3. *Suppose that Assumptions 1–2 hold and the first-layer subproblem (12a) is feasible in $X^{mpc}(N)$ at time $k = 0$. Then the system (1) in closed-loop with the controller obtained by Algorithm 1 is asymptotically stable with the region of attraction $X^{mpc}(N)$.*

Proof. Due to the initial feasibility of the first-layer subproblem (12a) in set $X^{mpc}(N)$, it is obtained from Theorems 1 and 2 that the lexicographic problem (12a)–(12c) is feasible at each time $k \in \mathbf{I}_{\geq 0}$ in $X^{mpc}(N)$.

Let $\mathbf{u}^*_{k,N}$ and $\mathbf{u}^{1*}_{k,N}$ be the optimal solutions to the whole problem (12) and the first-layer subproblem (12a) at time k, respectively. In general, $\mathbf{u}^*_{k,N} \neq \mathbf{u}^{1*}_{k,N}$. As $\mathbf{u}^*_{k,N}$ is an optimal solution to the whole problem (12), we have $J_1(\mathbf{u}^*_{k,N}, x_k) \leq J^*_1(x_k) = J_1(\mathbf{u}^{1*}_{k,N}, x_k)$. Select the sequence (13) as a feasible solution to the first-layer subproblem (12a) at time $k+1$. It is obtained that $J_1(\mathbf{u}^*_{k+1,N}, x_{k+1}) \leq J_1(\mathbf{u}^{1*}_{k+1,N}, x_{k+1}) \leq J_1(\tilde{\mathbf{u}}^1_{k+1,N}, x_{k+1})$, where $\mathbf{u}^*_{k+1,N}$ and $\mathbf{u}^{1*}_{k+1,N}$ are the optimal solutions to the whole problem (12) and the first-layer subproblem (12a) at time $k+1$, respectively.

For the value functions $J_1(\mathbf{u}^*_{k+1,N}, x_{k+1})$ and $J_1(\mathbf{u}^*_{k,N}, x_k)$ at time k and $k+1$, respectively, we derive that

$$\begin{aligned} & J_1(\mathbf{u}^*_{k+1,N}, x_{k+1}) - J_1(\mathbf{u}^*_{k,N}, x_k) \\ & \quad \leq J_1(\tilde{\mathbf{u}}^1_{k+1,N}, x_{k+1}) - J_1(\mathbf{u}^*_{k,N}, x_k) \\ & \quad = E_1(x_{N+1|k}) - E_1(x_{N|k}) + L_1(x_{N|k}, \kappa_1^{loc}(x_{N|k})) \\ & \qquad - L_1(x_{0|k}, u^*_{0|k}). \end{aligned} \tag{14}$$

From Assumption 2, the inequality (14) yields

$$J_1(\mathbf{u}^*_{k+1,N}, x_{k+1}) - J_1(\mathbf{u}^*_{k,N}, x_k) \leq -L_1(x_{0|k}, u^*_{0|k}) < 0. \tag{15}$$

From Assumption 1, the stage cost $L_1(x, u)$ and the terminal cost $E_1(x)$ are continuous, positive definite functions. This implies that $J_1(\mathbf{u}^*, x)$ is positive definite and has a lower limit. From the Lyapunov's augments (Khalil, 2002), $J_1(\mathbf{u}^*, x)$ is then a Lyapunov function of the system (1) in closed-loop with the controller obtained by Algorithm 1. This leads to the asymptotical stability of the equilibrium point of the closed-loop system. Moreover, since the terminal region Ω is an invariant set, the set $X^{mpc}(N)$ is a region of attraction of the closed-loop system (Rawlings & Mayne, 2009). ■

Remark 2. From the proof of Theorem 3 and Assumption 1, it is known that the stability of the lexicographic NMPC is only determined by the first objective function, i.e., the most important objective function. This implies that the other objective functions are not necessarily positive definite but may be economic functions. At the same time, the value functions $J_i(\mathbf{u}^*, x)$ of the ith-layer subproblem for all $i \in \mathbf{I}_{2:l}$ are not necessarily a Lyapunov function of the closed-loop system. Hence, the stability of the lexicographic NMPC is decoupled with the ith-layer subproblem for all $i \in \mathbf{I}_{2:l}$. This will benefit such cases that several objective functions with lower priorities have to be given up due to missing data or actuator faults, while maintaining the stability of the closed-loop system.

Remark 3. The lexicographic MO-NMPC proposed here along with the results of the economic NMPC (see, e.g., Amrit, Rawlings, & Angeli, 2011, Angeli, Amrit, & Rawlings, 2012 and Grüne, 2013, etc.) can deal with the case where the most important objective function is not necessarily positive definite but may be economic. In particular, we consider the system (1) that is strictly dissipative with respect to the supply rate $s(x,u) = L_1(x,u) - L_1(x_s,u_s)$ and a storage function $\lambda : X \to R$, where the equilibrium point (x_s,u_s) is a steady-state optimal economic point (Angeli et al., 2012). Note that the cases of $L_1(x_s,u_s) = 0$ and $E_1(x_s) = 0$ may not hold in case of economic optimization. To this end, we define the rotated regulator cost functions, i.e.,

$$\bar{L}_1(x,u) = L_1(x,u) + \lambda(x) - \lambda(f(x,u)) - L_1(x_s,u_s), \tag{16}$$

$$\bar{E}_1(x,u) = E_1(x) + \lambda(x) - E_1(x_s) - \lambda(x_s), \tag{17}$$

$$\bar{J}_1(\mathbf{u}_{k,N}, x_k) = \bar{E}_1(x_{N|k}) + \sum_{t=0}^{N-1} \bar{L}_1(x_{t|k}, u_{t|k}), \tag{18}$$

and revise Assumption 2, i.e., there exist an invariant set $\Omega \subseteq X$ of system (1), containing the origin in its interior, and a local control law $u = \kappa_1^{loc}(x)$ such that

$$\begin{aligned} &\kappa_1^{loc}(x) \in U, \qquad f(x,\kappa_1^{loc}(x)) \in \Omega, \\ &E_1(f(x,\kappa_1^{loc}(x))) - E_1(x) + L_1(x,\kappa_1^{loc}(x)) \le L_1(x_s,u_s) \end{aligned} \tag{19}$$

for any $x \in \Omega$. By using the similar procedure presented by Amrit et al. (2011), it can be shown that the optimal value function of (21) is a Lyapunov function of the closed-loop system and hence, the steady-state point (x_s,u_s) is an asymptotically equilibrium point of the closed-loop system with a region of attraction $X^{mpc}(N)$.

Remark 4. In some applications, the priorities of objectives may vary in time due to the change of working condition. The change in objective prioritization yields the results of switching control, which are similar to the results by Magni et al. (2008) and Müller and Allgöwer (2012). From the switching control theory (Liberzon, 2003), the stability results (e.g., feasibility and stability) obtained in case of invariant objective prioritization may not hold when the priorities of the objectives vary in time. In order to regain the feasibility of the MO-NMPC problem in case of varying objective prioritization, one method is to impose a new assumption of the objective functions to be optimized, i.e.,

Assumption 3. The stage cost $L_i(x,u)$ and terminal cost $E_i(x)$ are continuous and positive definite in their augments for all $i \in \mathbf{I}_{1:l}$. Moreover, there exist an invariant set $\Omega \subseteq X$ of system (1), containing the origin in its interior, and a local control law $u = \kappa_i^{loc}(x)$ such that

$$\begin{aligned} &\kappa_i^{loc}(x) \in U, \qquad f(x,\kappa_i^{loc}(x)) \in \Omega, \\ &E_i(f(x,\kappa_i^{loc}(x))) - E_i(x) + L_i(x,\kappa_i^{loc}(x)) \le 0. \end{aligned} \tag{20}$$

for any $x \in \Omega$ and all $i \in \mathbf{I}_{1:l}$. In other words, the invariant set Ω is a common sublevel set of the terminal cost $E_i(x)$ for all $i \in \mathbf{I}_{1:l}$.

Under Assumption 3, the feasibility of the MO-NMPC problem with varying objective prioritization is achieved by combining Theorem 1 and the similar procedure for recursive feasibility presented by Müller and Allgöwer (2012). Note that the invariant set Ω satisfying (20) might be strongly conservative in terms of its size. However, the conditions for the stability of the MO-NMPC are not obtained in case of varying objective prioritization. One possible way to achieve this stability of the MO-NMPC is to impose dwell time (Liberzon, 2003) for the objective priority change. The stability and feasibility for more general MO-NMPC with changing priority will be pursued in our future work.

4. Suboptimal prioritized MO-NMPC

In the lexicographic FHOCP (12), the number of constraints is increased after solving each single objective optimization subproblem. This leads to the computational load of the whole problem to be more than the sum of loads of the individual objective problem. To reduce the computational load, we attempt to decrease the number of layers of the whole problem (12) by such methods as uniting the objectives with identical priorities. On the other hand, global solutions of each layer subproblem generally cannot be guaranteed or are highly expensive computationally for nonlinear and non-convex programming. From the viewpoint of application, hence, one approach of particular interest is to design a suboptimal version of Algorithm 1. It should be emphasized that the suboptimality is defined here in terms of each layer subproblem. In this section, a suboptimal prioritized MO-NMPC algorithm is presented to guarantee the feasibility of the lexicographic FHOCP and the stability of the obtained suboptimal MO-NMPC.

Now we define a suboptimal lexicographic FHOCP, i.e., finding

$$\tilde{\mathbf{u}}_{k,N}^1 \in \left\{ \mathbf{u}_{k,N} \in U^N \,\middle|\, \begin{array}{l} J_1(\mathbf{u}_{k,N}, x_k) \le J_1(\tilde{\mathbf{u}}_{k,N}^0, x_k), \\ \text{(5b)}, \ x_{N|k} \in \Omega \end{array} \right\} \tag{21a}$$

$$\tilde{\mathbf{u}}_{k,N}^2 \in \left\{ \mathbf{u}_{k,N} \in U^N \,\middle|\, \begin{array}{l} J_2(\mathbf{u}_{k,N}, x_k) \le J_2(\tilde{\mathbf{u}}_{k,N}^1, x_k), \\ J_1(\mathbf{u}_{k,N}, x_k) \le J_1(\tilde{\mathbf{u}}_{k,N}^1, x_k), \\ \text{(5b)}, \ x_{N|k} \in \Omega \end{array} \right\} \tag{21b}$$

$$\vdots$$

$$\tilde{\mathbf{u}}_{k,N}^i \in \left\{ \mathbf{u}_{k,N} \in U^N \,\middle|\, \begin{array}{l} J_i(\mathbf{u}_{k,N}, x_k) \le J_i(\tilde{\mathbf{u}}_{k,N}^{i-1}, x_k), \\ J_j(\mathbf{u}_{k,N}, x_k) \le J_j(\tilde{\mathbf{u}}_{k,N}^{j}, x_k), \\ \text{(5b)}, \ x_{N|k} \in \Omega, \ \forall j \in \mathbf{I}_{1:i-1} \end{array} \right\} \tag{21c}$$

for all $i \in \mathbf{I}_{2:l}$, where the sequence $\tilde{\mathbf{u}}_{k,N}^0$ is an initial feasible solution satisfying (5b) and terminal constraint $x_{N|k} \in \Omega$, and $\tilde{\mathbf{u}}_{k,N}^i$ denotes a suboptimal solution to the ith-layer subproblem for $i \in \mathbf{I}_{1:l}$. Then a suboptimal version of Algorithm 1 is described by Algorithm 2.

Algorithm 2 (*Suboptimal MO-NMPC Algorithm*).

(1) Input the objective functions $J_i(\mathbf{u}_{k,N}, x_k)$ in (4) and their priorities; set $k = 0$ and pick a sequence $\tilde{\mathbf{u}}_{0,N}^0$ satisfying (5b) and terminal constraint $x_{N|0} \in \Omega$ with initial state x_0.
(2) Measure the state x_k at time step k.
(3) Solve the ith-layer subproblem of (21) for all $i \in \mathbf{I}_{1:l}$ and find a suboptimal sequence $\tilde{\mathbf{u}}_{k,N}^i$.
(4) Determine a lexicographic suboptimal sequence of the whole problem (21), $\tilde{\mathbf{u}}_{k,N} = \tilde{\mathbf{u}}_{k,N}^l$, and apply the first element $u_k^{mpc} = \tilde{u}_{0|k}$ to the system (1).
(5) Set $k = k + 1$ and find a control sequence $\tilde{\mathbf{u}}_{k,N}^0$ that satisfies (5b), terminal constraint $x_{N|k} \in \Omega$ and

$$J_1(\tilde{\mathbf{u}}_{k,N}^0, x_k) \le J_1(\tilde{\mathbf{u}}_{k-1,N}, x_{k-1}). \tag{22}$$

(6) Go to step (2). ■

The lexicographic suboptimal MO-NMPC by Algorithm 2 has the similar the feasibility and stability results to the lexicographic optimal MO-NMPC by Algorithm 1.

Theorem 4. *Under Assumptions 1–2, the lexicographic suboptimal FHOCP (21a)–(21c) admits the hierarchical feasibility at each time $k \in \mathbf{I}_{\ge 0}$.*

Proof. The proof procedure is close to that of Theorem 1 and then is omitted here. ■

Theorem 5. *Suppose that Assumptions 1–2 hold and there is a solution $\tilde{\mathbf{u}}^0_{k,N}$ that satisfies constraints (5b) and $x_{N|k} \in \Omega$ at time $k \in \mathbf{I}_{\geq 0}$. Then the lexicographic suboptimal FHOCP (21a)–(21c) admits the horizontal feasibility.*

Proof. Considering the first-layer subproblem (21a) at time $k \in \mathbf{I}_{\geq 0}$, it is known clearly that the sequence $\tilde{\mathbf{u}}^0_{k,N}$ is its feasible solution. Setting $i = 1$ and by induction, it is derived from Theorem 4 that the whole suboptimal problem (21) is feasible at time k.

Let $\tilde{\mathbf{u}}_{k,N} = \{\tilde{u}_{0|k}, \tilde{u}_{1|k}, \ldots, \tilde{u}_{N-1|k}\}$ be a suboptimal solution to the whole problem (21) at time k. From Algorithm 2, we have $\tilde{\mathbf{u}}_{k,N} = \tilde{\mathbf{u}}^l_{k,N}$, where $\tilde{\mathbf{u}}^l_{k,N}$ is a suboptimal solution to the lth-layer subproblem of (21). In order to find a feasible solution to the whole problem (21) at next time $k+1$, we consider the following control sequence:

$$\tilde{\mathbf{u}}^0_{k+1,N} = \{\tilde{u}^l_{1|k}, \ldots, \tilde{u}^l_{N-1|k}, \kappa^{loc}_1(x_{N|k})\} \tag{23}$$

where control law $\kappa^{loc}_1(x)$ satisfies Assumption 2. Since the terminal state $x_{N|k} \in \Omega$, it is known from Assumption 2 that the control action $\kappa^{loc}_1(x_{N|k}) \in U$ and the terminal state $x_{N|k+1} \in \Omega$. This suggests that the sequence (23) satisfies the constraints (5b) and terminal constraint $x_{N|k+1} \in \Omega$. Moreover, considering the value functions $J_1(\tilde{\mathbf{u}}^0_{k+1,N}, x_{k+1})$ and $J_1(\tilde{\mathbf{u}}_{k,N}, x_k)$ at time k and $k+1$, respectively, we have

$$\begin{aligned} J_1(\tilde{\mathbf{u}}^0_{k+1,N}, x_{k+1}) &- J_1(\tilde{\mathbf{u}}_{k,N} x_k) \\ &= J_1(\tilde{\mathbf{u}}^0_{k+1,N}, x_{k+1}) - J_1(\tilde{\mathbf{u}}^l_{k,N}, x_k) \\ &= E_1(x_{N+1|k}) - E_1(x_{N|k}) + L_1(x_{N|k}, \kappa^{loc}_1(x_{N|k})) \\ &\quad - L_1(x_{0|k}, \tilde{u}_{0|k}). \end{aligned} \tag{24}$$

From Assumption 2, the inequality (22) holds. Hence, the first-layer subproblem (21a) has a feasible solution, i.e., (23), at time $k+1$. Then by induction and applying Theorem 4 again, it is obtained that the lexicographic suboptimal FHOCP (21a)–(21c) is feasible at time $k+1$, i.e., the whole problem (21) admits the horizontal feasibility. ■

Remark 5. The proof of Theorem 5 implies that the sequence (23) gives a candidate of the initial feasible solution $\tilde{\mathbf{u}}^0_{k,N}$ at each time $k \in \mathbf{I}_{\geq 0}$, which can be used as warm starting to decrease the computational demand of solving the first-layer subproblem at the next time.

Theorem 6. *Suppose that Assumptions 1–2 hold and there is an initial solution $\tilde{\mathbf{u}}^0_{k,N}$ in $X^{mpc}(N)$ at time $k = 0$. Then the system (1) in closed-loop with the controller obtained by Algorithm 2 is asymptotically stable with the region of attraction $X^{mpc}(N)$.*

Proof. Due to the existence of the initial solution $\tilde{\mathbf{u}}^0_{k,N}$ in $X^{mpc}(N)$ at time $k = 0$, it is derived from Theorems 4 and 5 that the lexicographic suboptimal problem (21a)–(21c) is feasible in $X^{mpc}(N)$ at each time $k \in \mathbf{I}_{\geq 0}$.

Let $\tilde{\mathbf{u}}_{k,N}$ and $\tilde{\mathbf{u}}^1_{k,N}$ be a suboptimal solution to the whole problem (21) and the first-layer subproblem (21a) at time k, respectively. Note that $\tilde{\mathbf{u}}_{k,N} \neq \tilde{\mathbf{u}}^1_{k,N}$, in general. As $\tilde{\mathbf{u}}_{k,N}$ is a suboptimal solution to the whole problem (21), we have $J_1(\tilde{\mathbf{u}}_{k,N}, x_k) \leq J_1(\tilde{\mathbf{u}}^1_{k,N}, x_k)$. Select the sequence (23) as an initial feasible solution to the first-layer subproblem (21a) at time $k+1$. We have $J_1(\tilde{\mathbf{u}}_{k+1,N}, x_{k+1}) \leq J_1(\tilde{\mathbf{u}}^1_{k+1,N}, x_{k+1}) \leq J_1(\tilde{\mathbf{u}}^0_{k+1,N}, x_{k+1})$, where $\tilde{\mathbf{u}}_{k+1,N}$ and $\tilde{\mathbf{u}}^1_{k+1,N}$ are the suboptimal solutions to the whole problem (21) and the first-layer subproblem (21a) at time $k+1$, respectively.

Table 1
Two cases of the objective prioritization.

	High	Middle	Low
Case 1	$J_1(\mathbf{u}, x)$	$J_2(\mathbf{u}, x)$	$J_3(\mathbf{u}, x)$
Case 2	$J_2(\mathbf{u}, x)$	$J_1(\mathbf{u}, x)$	$J_3(\mathbf{u}, x)$

Consider the value functions $J_1(\tilde{\mathbf{u}}_{k+1,N}, x_{k+1})$ and $J_1(\tilde{\mathbf{u}}_{k,N}, x_k)$ at time k and $k+1$, respectively. From Assumption 2 and the inequality (24), we derive that

$$\begin{aligned} J_1(\tilde{\mathbf{u}}_{k+1,N} x_{k+1}) - J_1(\tilde{\mathbf{u}}_{k,N}, x_k) &\leq J_1(\tilde{\mathbf{u}}^0_{k+1,N}, x_{k+1}) - J_1(\tilde{\mathbf{u}}_{k,N} x_k) \\ &\leq -L_1(x_{0|k}, \tilde{u}_{0|k}) < 0. \end{aligned} \tag{25}$$

Using the similar procedure of the proof of Theorem 3, then we have the conclusion that the equilibrium point of the closed-loop system is asymptotically stable in the region of attraction $X^{mpc}(N)$. ■

Remark 6. In single objective MPC, one of well-known conclusions is that "feasibility implies stability" (Rawlings & Mayne, 2009; Scokaert et al., 1999). Theorem 6 suggests that this conclusion is also true in case of multiobjective optimization given by Algorithm 2. As argued by Scokaert et al. (1999), a benefit of this conclusion is that we can use early termination to decrease the computational demand of solving each layer subproblem while obtaining a stabilizing MO-NMPC controller.

5. Numerical example

To illustrate the effectiveness of the proposed results, we consider the nonlinear system described by

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} x_{2,k} \sin x_{1,k} \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_k. \tag{26}$$

Here the state and control variables are subject to the box constraints $|x_{i,k}| \leq 1$, $i = 1, 2$ and $|u_k| \leq 1$ for all time $k \in \mathbf{I}_{\geq 0}$, respectively. For this example, we assume that there are three different, prioritized cost functions to be minimized at the each time, i.e.,

$$J_1(\mathbf{u}_{k,N}, x_k) = \|P_1 x_{N|k}\|_\infty + \sum_{t=0}^{N-1} (\|Q_1 x_{t|k}\|_\infty + \|R_1 u_{t|k}\|_\infty) \tag{27}$$

$$J_2(\mathbf{u}_{k,N}, x_k) = x^T_{N|k} P_2 x_{N|k} + \sum_{t=0}^{N-1} (x^T_{t|k} Q_2 x_{t|k} + u^T_{t|k} R_2 u_{t|k}) \tag{28}$$

$$J_3(\mathbf{u}_{k,N}, x_k) = \sum_{t=0}^{N-1} (u_{t|k} - u_{t-1|k})^T (u_{t|k} - u_{t-1|k}) \tag{29}$$

where $u_{-1|k} = u_{k-1}$ and weighted matrices

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad R_1 = 0.1, \quad P_1 = \begin{bmatrix} 9.6 & 1.4 \\ -0.3 & 9.4 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = 0.2, \quad P_2 = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 1.3 \end{bmatrix}.$$

Note that the terminal costs $E_1(x) = \|P_1 x\|_\infty$ and $E_2(x) = x^T P_2 x$ are positive definite functions, but the terminal cost $E_3(x)$ in $J_3(\mathbf{u}, x)$ is not a positive definite function as $E_3(x) = 0$ for all $x \in X$.

The goal of this example is to design an MO-NMPC that simultaneously minimizes the objective functions (27)–(29) according to a given objective prioritization and meanwhile stabilizes the system (26) to the origin in the presence of the box constraints. Here we consider two cases of the objective prioritization given in Table 1.

To compare the results achieved here, the utopia-tracking MO-NMPC scheme (e.g. Zavala & Flores-Tlacuahuac, 2012) is used. Let

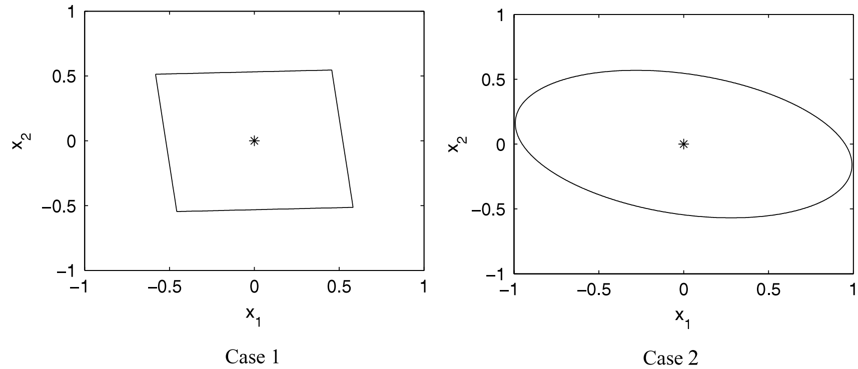

Fig. 1. The terminal region Ω_1 (left) and Ω_2 (right).

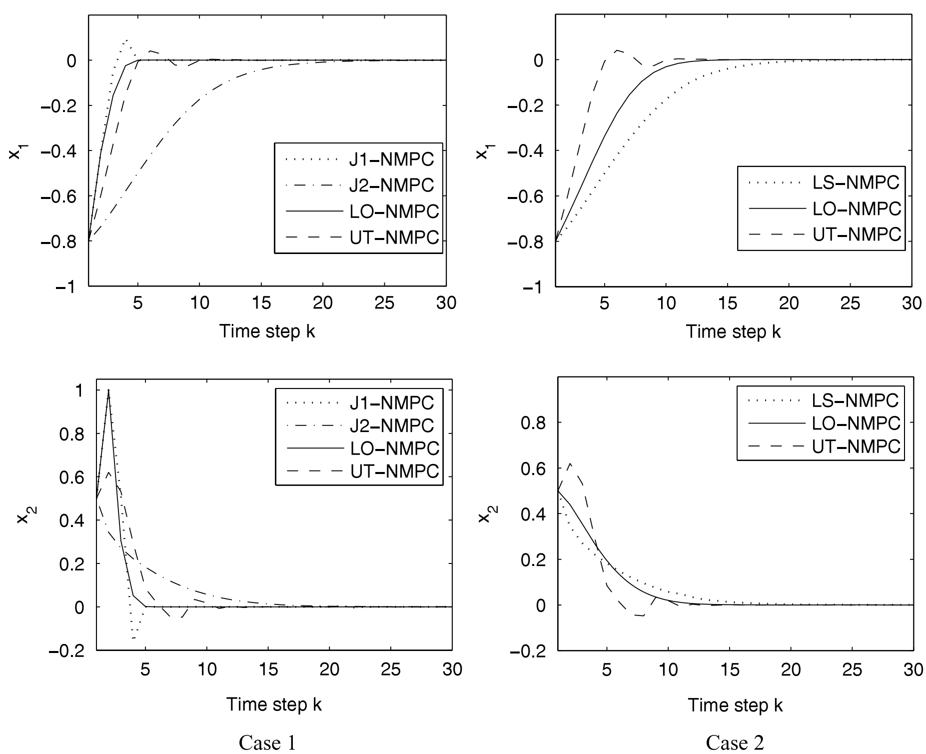

Fig. 2. The closed-loop state trajectories.

UT-NMPC and LO-NMPC denote the utopia-tracking MO-NMPC and the lexicographic optimal MO-NMPC obtained by Algorithm 1, respectively. For both controllers, all FHOCPs are solved by the MatLab function 'fmincon' using MATLAB 7.1 under Windows 7 and an Intel(R) Core(TM) 2 Duo CPU with 2.2 GHz and 2 GB RAM. The solution at time k is used to be an initial guess for solving the FHOCP at next time $k+1$ to improve the computational efficiency of NMPC controllers. Let the prediction horizon $N = 10$ and the small tolerance $\varepsilon_i = 0.05J_i^*$. Pick an initial state $x_0 = [-0.8, 0.5]^T$ for all simulation experiments.

Case 1: In this case, decision-makers first focus on the l_∞-norm of weighted states over a time interval, then system transition and finally control increments. Consider the most important objective $J_1(\mathbf{u}, x)$. We compute the local control law $u = \kappa_1^{loc}(x)$ and terminal region Ω_1 to satisfy Assumption 2. By the linearized system of (26) at the origin, a local control law is obtained as Bemporad and Munoz de la Pena (2009) and Lazar et al. (2006)

$$\kappa_1^{loc}(x) = -\begin{bmatrix}0.5 & 1.4\end{bmatrix} x. \tag{30}$$

Then we calculate the terminal region

$$\Omega_1 = \{x \in R^2 : \|P_1 x\|_\infty \le 5.01\} \tag{31}$$

such that it is invariant for the system (26) in closed-loop with $u = \kappa_1^{loc}(x)$. The region Ω_1 is shown by the polytope in the left graph of Fig. 1.

The left graphs of Figs. 2 and 3 demonstrate the different responses of system (26) driven by LO-NMPC (solid), UT-NMPC (dash) and SO-NMPC minimizing J_1 (dotted, denoted by J1-NMPC) and minimizing J_2 (dash-dotted, denoted by J2-NMPC), respectively. Clearly, J1-NMPC minimizing the l_∞-norm of weighted states, leads to more oscillatory state responses and J2-NMPC results in a slow transition. In contrary, LO-NMPC and UT-NMPC consider the objective functions (27)–(29) simultaneously and aim to improve the responses of the closed-loop system in terms of the magnitudes of signals and the transition time. In particular, LO-NMPC generates smaller state fluctuation and faster transition than other algorithms compared, which properly reflects decision makers' control goal.

The left graphs of Fig. 4 depict the value profiles of cost functions (27)–(29) derived by LO-NMPC (solid), UT-NMPC (dash), J1-NMPC (dotted) and J2-NMPC (dash-dotted), respectively. It is seen that J1-NMPC and J2-NMPC only minimize their individual cost function, but not the others. Moreover, the values of the cost functions

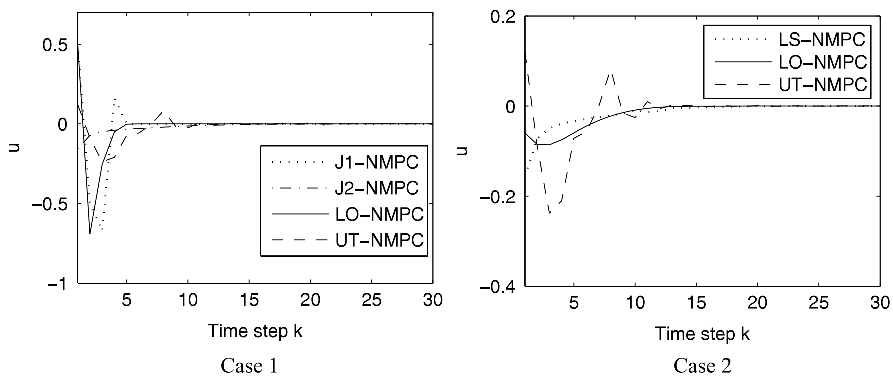

Fig. 3. The control input profiles.

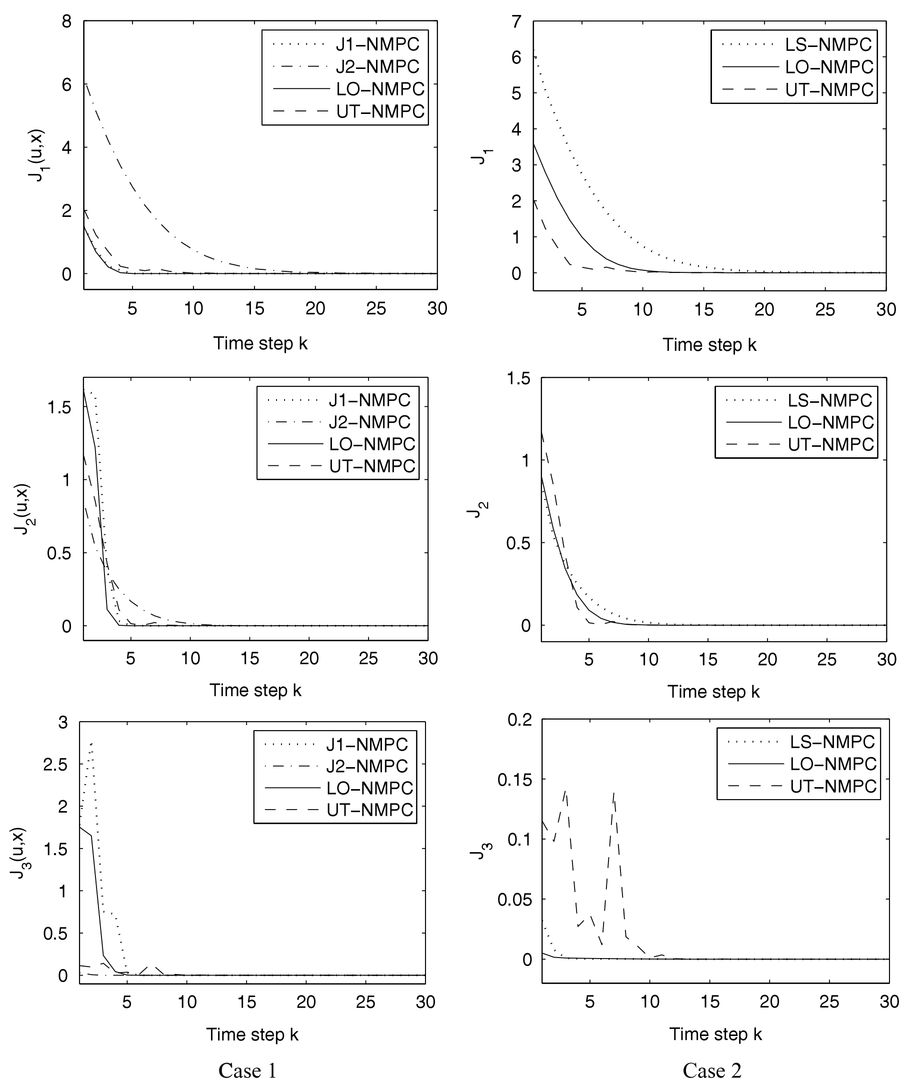

Fig. 4. The value profiles of the cost functions.

achieved by UT-NMPC lie between those by J1-NMPC and J2-NMPC. Finally, considering the objective prioritization in case 1, we observe that the values of J_1 achieved by LO-NMPC are equal to those by J1-NMPC and the values of J_2 and J_3 by LO-NMPC are less than those by J1-NMPC. Note that due to the lower priorities of J_2 and J_3, the values of J_2 and J_3 reached by LO-NMPC are not necessarily less than those by J2-NMPC. In fact, the values of J_2 reached by LO-NMPC are less than those by the others only after three time steps, which suggests that the value profiles derived by LO-NMPC do not inside the 'tube' defined by the value profiles derived respectively by J1-NMPC and J2-NMPC.

Case 2: In this case, the most important objective function is (28), i.e., $J_2(\mathbf{u}, x)$. Hence, decision-makers first emphasize system transient performance, then the l_∞-norm of weighted states over a time interval and finally control increments. By solving the LQR problem of the linearized system of (26) at the origin, we have a

Table 2
Average time taken to solve the NMPC problems.

	UT-NMPC (ms)	LO-NMPC (ms)	LS-NMPC (ms)
$N = 5$	88	140	5.6
$N = 10$	147	260	9.1
$N = 20$	759	992	16.2

local control law and a terminal region

$$\kappa_2^{loc}(x) = -\begin{bmatrix}0.2 & 1.0\end{bmatrix} x, \\ \Omega_2 = \{x \in R^2 : x^T P_2 x \le 0.39\} \tag{32}$$

which satisfy Assumption 2 with respect to J_2. The terminal region is shown by the ellipsoid in the right graph of Fig. 1.

The right graphs of Figs. 2 and 3 show the different responses of system (26) driven by LO-NMPC (solid), UT-NMPC (dash) and LS-NMPC (dotted), respectively. Here LS-NMPC denotes the lexicographic suboptimal MO-NMPC obtained by Algorithm 2. Obviously, UT-NMPC generates the same responses for different objective prioritizations. However, LO-NMPC designed in case 2 have different responses from those achieved in case 1 (see the left graphs of Figs. 2 and 3). The reason of these differences is that the LO-NMPC and LS-NMPC depend on the objective prioritization. Moreover, comparing the dash-dotted lines in the left graphs of Fig. 2 to the solid lines in the right graphs of Fig. 2, it is known that the system in closed-loop with LO-NMPC in case 2 have better responses than in closed-loop with J2-NMPC.

The right graphs of Fig. 4 show the value profiles of the three cost functions in case 2, derived by LO-NMPC (solid), UT-NMPC (dash) and LS-NMPC (dotted), respectively. We note that for the objective prioritizations in case 2, UT-NMPC generates the same cost functions values as those in case 1 (see the left graphs of Fig. 4). On the contrary, LO-NMPC first guarantees the minimization of J_2 and then tackles the minimization of the rest, which is shown by the solid lines in the right graphs. This implies that LO-NMPC is designed by taking into account the objective prioritization. Compared to the LO-NMPC, the LS-NMPC has an increase in terms of cost functions values, which is the price one has to pay in order to improve online computation efficiency of solving FHOCPs (see Table 2).

In addition, in order to compare the computational demand of the algorithms, Table 2 tabulates the average computational time taken to solve the NMPC problems at one time step applied by UT-NMPC, LO-NMPC and LS-NMPC. In LS-NMPC, the two-division method is used to compute a suboptimal control sequence of FHOCP (21a)–(21c). It is observed that the computational demand of LO-NMPC is almost two times that of UT-NMPC, which is caused mainly by the fact that LO-NMPC simultaneously minimizes three single objective optimization problems at each time. However, there is a substantial reduction in computational demand of LS-NMPC by Algorithm 2 due to simplifying the problem from finding an optimal solution to searching a feasible solution. Note that this computational reduction is achieved at the expense of the optimality of performance, i.e., the loss of optimality, which is demonstrated by the solid and dotted lines in right graphs of Fig. 4. But the complex interplay between the computational reduction and the performance variation is still an open issue to be pursued in our future work.

6. Conclusions

In this paper we proposed a lexicographic MO-NMPC scheme to solve prioritized multiobjective optimal control problems of constrained nonlinear systems. The feature of this scheme is that it takes explicitly into account the objective prioritization during the period of designing controllers, which provides a systematic method to balance multiple competing criteria without using weighting matrices. The conditions for guaranteeing feasibility and stability of the MO-NMPC are determined only by the most important objective function. Hence, this scheme can deal with some economic objectives. A suboptimal MO-NMPC algorithm was presented to reduce the computational load for solving the optimization problem and the property that feasibility implies stability in single objective NMPC was regained in MO-NMPC. The comparison results of the example demonstrated the performance and effectiveness of the scheme proposed in the paper.

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