Internal model control (IMC), which explicitly incorporates a plant model and a plant inverse model as its components, has an intuitive control structure and simple tuning procedure. Within the IMC structure, we propose composite adaptive IMC (CAIMC) which simultaneously identifies the plant and the plant inverse to minimize modeling errors and further reduce the tracking error. In this paper, the design procedure of CAIMC is generalized to an $n$-th-order SISO plant. The main challenge in the generalization is to find an identification algorithm for an $n$-th order system that satisfies the stability constraint, while assuring closed-loop stability. In the literature, stability-constrained identification has been formulated as a convex programming problem by re-parameterizing the constraint as a linear matrix inequality, but boundedness and continuity of the estimated parameters, which are critical for closed-loop stability of an adaptive control algorithm, are not guaranteed. We propose a modified stability-constrained identification method with established boundedness and continuity properties. Closed-loop stability and asymptotic performance of CAIMC are then established under proper conditions. The effectiveness of the proposed algorithm is demonstrated with an example.

**Index Terms**—Adaptive control, LMIs, system identification, optimization.

**NOMENCLATURE**

- $R$, $R_i$ Real numbers, $i \times 1$ real vectors.
- $I_i$, $O_{i \times j}$ $i \times i$ identity matrix, $i \times j$ zero matrix.
- $[\lambda^T, \lambda]$ Transpose, eigenvalues of a matrix.
- $\text{vec}(\cdot)$ A column vector from vectorization of a matrix.
- $e, e_M, e_Q$ Tracking error, modeling error, and inverse modeling error.
- $k, m, n$ Integers.
- $l$ IMC feedback, $l = r - y + y_M$.
- $r$ Reference.
- $u, y, y_M$ Plant input, plant output, model output.

**I. INTRODUCTION**

Internal model control (IMC) is an intuitive control structure with a simple tuning procedure [1]. As shown in Fig. 1, IMC incorporates the plant model $M$ as an explicit part of the controller. The controller $Q$ can be chosen as the approximate inverse of $M$, augmented with a filter to make it causal [2]–[5]. Combining the IMC structure with adaptive control leads to adaptive IMC (AIMC), which can handle unknown or slowly varying parameters in the plant and its operating environment. For standard AIMC, the plant is identified and the inverse is derived by inverting the estimated plant. Comprehensive studies on AIMC have been carried out in [6]–[9], and there exist many successful applications [10]–[12]. Simultaneous identification of the plant and the plant inverse is very tempting: intuitively, IMC performs better with more accurate plant and plant inverse models. By identifying the plant inverse directly, as opposed to calculating the inverse model from the identified plant model, one can have a more accurate representation of the plant inverse dynamics. This motivated composite AIMC (CAIMC) for a first-order plant as presented in [13], [14], where “composite” means that the plant and the plant inverse are identified simultaneously, is shown in Fig. 2. In this paper, CAIMC is generalized for $n$-th-order SISO plants.

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**Abstract**—Internal model control (IMC), which explicitly incorporates a plant model and a plant inverse model as its components, has an intuitive control structure and simple tuning procedure. Within the IMC structure, we propose composite adaptive IMC (CAIMC) which simultaneously identifies the plant and the plant inverse to minimize modeling errors and further reduce the tracking error. In this paper, the design procedure of CAIMC is generalized to an $n$-th-order SISO plant. The main challenge in the generalization is to find an identification algorithm for an $n$-th order system that satisfies the stability constraint, while assuring closed-loop stability. In the literature, stability-constrained identification has been formulated as a convex programming problem by re-parameterizing the constraint as a linear matrix inequality, but boundedness and continuity of the estimated parameters, which are critical for closed-loop stability of an adaptive control algorithm, are not guaranteed. We propose a modified stability-constrained identification method with established boundedness and continuity properties. Closed-loop stability and asymptotic performance of CAIMC are then established under proper conditions. The effectiveness of the proposed algorithm is demonstrated with an example.

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Internal model control (IMC) is an intuitive control structure with a simple tuning procedure [1]. As shown in Fig. 1, IMC incorporates the plant model $M$ as an explicit part of the controller. The controller $Q$ can be chosen as the approximate inverse of $M$, augmented with a filter to make it causal [2]–[5]. Combining the IMC structure with adaptive control leads to adaptive IMC (AIMC), which can handle unknown or slowly varying parameters in the plant and its operating environment. For standard AIMC, the plant is identified and the inverse is derived by inverting the estimated plant. Comprehensive studies on AIMC have been carried out in [6]–[9], and there exist many successful applications [10]–[12]. Simultaneous identification of the plant and the plant inverse is very tempting: intuitively, IMC performs better with more accurate plant and plant inverse models. By identifying the plant inverse directly, as opposed to calculating the inverse model from the identified plant model, one can have a more accurate representation of the plant inverse dynamics. This motivated composite AIMC (CAIMC) for a first-order plant as presented in [13], [14], where “composite” means that the plant and the plant inverse are identified simultaneously, is shown in Fig. 2. In this paper, CAIMC is generalized for $n$-th-order SISO plants.
The contribution of this paper is twofold: First, we present a stability-constrained identification algorithm that adds a regularization term to the approach proposed in [19], [21]. Uniqueness, boundedness, and continuity properties of the optimal solutions are established, which do not exist in the literature. The technique is general, in the sense that it is applicable to any adaptive control algorithm where stability constraint should be considered, and closed-loop stability can be established. Second, CAIMC is generalized for $n$-th-order SISO plants using the proposed stability-constrained identification algorithm, and closed-loop stability and asymptotic performance are established. The generalized CAIMC was originally presented as a conference paper [22] with the stringent assumption that the plant had relative degree zero and the stability constraint was not handled. In this paper, both issues are addressed.

This paper is organized as follows: In Section II, the standard identification technique is introduced and the stability-constrained identification problem is formulated as a convex programming problem based on techniques from [19] and [21]. Then, the modified stability-constrained identification is presented with a simulation, and properties for the identified parameters are established. In Section III, the design process of CAIMC for an $n$-th-order plant with relative degree $m \geq 0$ is presented. In Section IV, the closed-loop stability and asymptotic performance are established for the ideal case when there are no unmodeled dynamics, and the effect of unmodeled dynamics is discussed. In Section V, CAIMC is applied to a third-order linear time invariant (LTI) plant. Section VI draws the conclusions.

II. STABILITY-CONSTRAINED IDENTIFICATION

In this section, we present a stability-constrained identification algorithm that satisfies the requirements (R1)–(R4), and establish its boundedness and continuity properties. The preliminaries, including the linear parametric model and the unconstrained normalized adaptive law, are first introduced in Sections II-A and II-B. They are based on [15].

A. Linear Parametric Model

An $n$-th-order linear dynamic model with relative degree $m \geq 0$ can be assumed to have the general form

$$y^{(n)} + a_{n-1}^{*} y^{(n-1)} + \cdots + a_0 y = b_{n-m}^{*} u^{(n-m)} + b_{n-m-1}^{*} u^{(n-m-1)} + \cdots + b_0 u$$

(1)

where $u$ and $y$ are the plant input and output, respectively. Assume that the parameter vector $\theta^* = [b_0^*, b_1^*, \ldots, b_{n-m}^*, a_0^*, a_1^*, \ldots, a_{n-1}^*]^T$ is unknown, one can obtain a linear parametric model

$$z = \theta^T \phi,$$

(2)

where the signal $z$ and $\phi$ are defined as

$$z = \left\{ \frac{s^n}{\Lambda} \right\} y,$$

$$\phi = \left[ \frac{1}{\Lambda} u, \ldots, \left\{ \frac{s^{n-m}}{\Lambda} \right\} y, \frac{1}{\Lambda} y, \ldots, \left\{ \frac{s^{n-1}}{\Lambda} \right\} y \right]^T.$$
\(z\) and \(\phi\) are referred to as the observation and the regressor, respectively. Throughout the paper, \(\{\cdot\}\) represents the dynamic operator, whose transfer function is \((\cdot)\). \(A = s^n + \lambda_{n-1}s^{n-1} + \cdots + \lambda_0\) is chosen as a Hurwitz polynomial, \(\frac{1}{s}\) is introduced to avoid using derivatives in the identification. \(\theta^*\) can be estimated using standard adaptive laws, such as the least-squares algorithm or gradient method.

### B. Unconstrained Normalized Adaptive Law

The normalized gradient algorithm is presented here to obtain \(\theta\), the estimate of the unknown parameter vector \(\theta^*\) in (2). The algorithm identifies \(\theta\) by minimizing certain performance cost. For computational and robustness reasons, the identified parameters \(\theta(k)\) can be updated at specific time instants \(kT_s\), where \(T_s\) is the sampling time. Here, we consider a quadratic cost function of \(c: J(\theta) = \frac{\epsilon^Tm^2}{2}\), where

\[
e(t) = \frac{z(t) - \theta^T(k)\phi(t)}{m(t)^2}, \quad \forall t \in [kT_s,(k+1)T_s) \tag{3}
\]

is the normalized estimation error, and \(m(t)^2 = 1 + \phi(t)^T\phi(t)\) is the normalizing term. Applying the gradient method, one can obtain

\[
\theta(k + 1) = \theta(k) + \Gamma \int_{kT_s}^{(k+1)T_s} e(t)\phi(t)dt \tag{4}
\]

where \(\Gamma = \Gamma^T\) is a positive-definite matrix affecting how rapidly \(\theta\) converges.

**Lemma 1:** Let \(2 - T_s\lambda_{\max}(\Gamma) \geq c\) for some \(c > 0\). The adaptive law (4) for a parametric model (2) guarantees that

1. \(\theta \in \ell^\infty\),
2. \(\Delta \theta \in \ell^2\) where \(\Delta \theta(k) = \theta(k + 1) - \theta(k)\).
3. \(c, \epsilon, m \in \mathcal{L}_2 \cap \ell_{\infty}\),
4. If \(m, \phi \in \ell_{\infty}\) and \(\phi\) is persistently exciting (PE), then \(\theta(k) \rightarrow \theta^* \) as \(k \rightarrow \infty\) exponentially fast.

The normalized gradient algorithm (4) minimizes the cost function \(J(\theta)\) with no constraints, i.e., it allows \(\theta\) to lie anywhere in \(\mathbb{R}^{2n-m+1}\). In the closed-loop stability analysis of Section IV, one of the sufficient conditions for establishing closed-loop stability is the frozen-time stability of \(M\) and \(Q\), namely their denominators have to be Hurwitz at each sample time. Therefore, an algorithm to constrain the stability of an \(n\)-th order polynomial is required. As the order of the polynomial increases, however, the stability constraint becomes nonconvex with nonsmooth boundaries, which causes problems with standard techniques such as the projection algorithm. To handle the stability constraint, LMI is exploited.

### C. Stability-Constrained Identification

We seek Lyapunov inequality to represent the stability constraint: For plant (1) whose estimated transfer function has denominator \(s^n + \sum_{i=0}^{n-1} a_is^i\), the corresponding controllable canonical form of its state-space realization has \(A = \begin{bmatrix} -\hat{\theta}_A^T & I_{n-1} \end{bmatrix}\), where \(\hat{\theta}_A = [a_{n-1}, \ldots, a_0]^T \in \mathbb{R}^n\). The stability condition can be expressed as the nonemptiness of the set defined by \(P = \{P | P > 0, AP + PA^T < 0\}\).

Let \(\hat{\theta}_A = [a_{n-1}, \ldots, a_0]^T \in \mathbb{R}^n\) be the estimates of \([a_{n-1}, \ldots, a_0]^T\) from the gradient algorithm (4). In this section, an optimization problem is formulated to find a stable \(\hat{\theta}_A \in \mathbb{R}^n\) that best approximates the unconstrained parameter \(\theta_A\).

A natural formulation of the optimization problem is to minimize the quadratic error between \(\hat{\theta}_A\) and \(\hat{\theta}_A\), subject to \(\hat{\theta}_A \) satisfying the stability constraint:

\[
\min_{\hat{\theta}_A, P} \|\theta_A - \hat{\theta}_A\|_2^2, \quad \text{subject to } P > 0 \text{ and } \begin{bmatrix} -\theta_A^T & I_{n-1} \\ -I_{n-1} & 0 \\ I_{n-1} & 0 \\ 0 & -I \end{bmatrix} < 0. \tag{5}
\]

However, \(\theta_A^TP\) in (5) introduces a nonconvex bilinear matrix inequality (BMI). The BMI optimization problem (5) can be solved with global approaches such as branch and bound, but it is computationally expensive [20]. To reduce the computational complexity, the BMI optimization problem is reformulated as an LMI optimization problem.

#### 1) Stability-Constrained Identification in the Literature:

Using the technique as presented in [19] and [21], define a new variable

\[
H = \hat{\theta}_A^TP \in \mathbb{R}^n. \tag{6}
\]

A weighting matrix \(P\) is added to the quadratic cost function \(\|\theta_A - \hat{\theta}_A\|_2^2\), and (5) becomes

\[
\min_{\hat{\theta}_A, P} \|\theta_A^TP - H\|_2^2, \quad \text{subject to } P > 0 \text{ and } \begin{bmatrix} -H & I_{n-1} \\ -I_{n-1} & 0 \\ I_{n-1} & 0 \\ 0 & -I \end{bmatrix} < 0. \tag{7}
\]

Note that with the reformulated cost function in terms of the redefined parameters \(P\) and \(H\), (7) has an LMI constraint. While (7) is equivalent to (5), the new optimization formulation replaces the BMI with an LMI, leading to a simpler problem amenable to many effective solvers. Equation (7) satisfies the requirements (R1)–(R3) in the introduction; however, it does not satisfy requirement (R4). It has no established properties as reported in the literature, while boundedness and continuity of the estimated parameters are crucial for closed-loop stability of an adaptive control system. Also note that when the cost \(\|\theta_A^TP - H\|_2^2\) reaches its minimum, there may be infinitely many solutions for \(P\) and \(H\), which can be arbitrarily large. The indefiniteness may pose computational difficulties.

#### 2) Modified Stability-Constrained Identification:

For uniqueness and boundedness of the optimal solution \(P^*\) and \(H^*\), a regularizing term \(\gamma\|\langle \text{vec}(P), H \rangle\|_2^2\) is added to the cost function in (7) to make the cost strictly convex, where \(\gamma > 0\) is a small constant. The constraints in (7) are tightened by
\( P \succeq \epsilon_0 I \) and \( PA^T + AP \preceq -\epsilon_0 I \) to assure the existence of the optimal solution, where \( \epsilon_0 > 0 \) is a small constant. These modifications transform (7) to:

\[
\begin{aligned}
& \text{minimize} \quad \| \theta_A^T P - H \|^2_2 + \gamma \| \vec{(P, H)} \|_2^2 \\
& \text{subject to} \quad P \succeq \epsilon_0 I \quad \text{and} \\
& \begin{bmatrix}
-H \\
[I_{n-1} \ 0]P
\end{bmatrix} + \begin{bmatrix}
-H \\
[I_{n-1} \ 0]P
\end{bmatrix}^T \preceq -\epsilon_0 I. \quad (8)
\end{aligned}
\]

Solving the optimization problem returns the optimal \( P^* \) and \( H^* \). \( \hat{\theta}_A \) can be calculated as \( \hat{\theta}_A = (H^* P^{-1})^T \) according to (6). Equation (8) can be solved efficiently with interior point methods, for which there are many mature tools available [23].

Given the parametric model (2), the implementation of the constrained identification is summarized as follows:

Algorithm 1: At the \( k \)-th Sample:

1) Use the unconstrained adaptive law (4) to calculate \( \hat{\theta}(k) \).
2) Calculate the stability-constrained \( \bar{\theta}(k) \) from \( \hat{\theta}(k) \).
   - Solve the convex optimization problem (8) for the optimal solution \( P^* \) and \( H^* \), with \( \hat{\theta}_A = \theta_A(k) \), where \( \theta_A(k) \) is the parameter vector of the transfer function denominator parameters in \( \theta(k) \).
   - Compute the constrained parameter vector \( \bar{\theta}_A(k) = (H^* P^{-1})^T \).
   - \( \bar{\theta}(k) \), which is obtained by substituting \( \bar{\theta}_A(k) \) in the unconstrained parameters vector \( \theta(k) \) with \( \bar{\theta}_A(k) \), is used for the control signal calculation.

D. Modified Stability-Constrained Identification Simulation

This section demonstrates the effectiveness of Algorithm 1 for constraining the identified system in the stable region. Consider a third-order stable LTI plant

\[
y = \begin{bmatrix} \theta_{b2} s^2 + \theta_{a1} s + \theta_{a0} \\ s^3 + \theta_{b2} s^2 + \theta_{a1} s + \theta_{a0} \end{bmatrix} u \quad (9)
\]

where \( \theta_{b2} = [2, 2, 3]^T, \theta_{a1} = [2, 2, 1]^T \) are assumed to be unknown. A third-order plant is adopted because it has a non-convex stability region, and the standard projection algorithm does not apply to such a constraint. White noises of variance 1 are added to the plant input and output with a first-order noise model. The signal-to-noise ratio is 10. The time constant of the noise model is uniformly distributed between 0 and 0.5.

Algorithm 1 is applied to estimate the unknown parameters: at each sample time, \( \hat{\theta} \) is calculated from the unconstrained adaptive law (4), then \( \bar{\theta} \) is calculated from \( \hat{\theta} \) considering the stability constraints. Five thousand identifications are performed with different initial values and different noise model time constant. Each identification is performed with 100 sample times. The three poles of the final identified system are shown in Fig. 3.

The actual poles are marked by “+.” The estimated poles on the open left half plane (OLHP) are marked in green, and the ones on the right half plane (RHP) are marked in red. Algorithm 1 keeps the identified system poles in the stable region.

E. Modified Stability-Constrained Identification Analysis

The continuity and boundedness properties of the identified parameter \( \hat{\theta} \) are crucial for closed-loop stability of the adaptive control system. In this section, we establish these properties by using tools from the optimization field to analyze the optimization problem (8).

When \( \gamma > 0 \), the cost function in (8) is strictly convex. Since the feasible set of (8) is nonempty, closed, and convex, from Lemma 5 in Appendix A, there exists a unique optimal solution. Let \( f(\chi, \theta_A, \gamma) \) represent the cost function of (8), where \( \chi = [\vec{(P, H)}] \). Let \( P^*(\theta_A, \gamma) \) and \( H^*(\theta_A, \gamma) \) represent the optimal solution. \( \hat{\theta}_A(\theta_A, \gamma) = (H^*(\theta_A, \gamma) P^*(\theta_A, \gamma)^{-1})^T \) is the constrained parameter vector.

Lemma 2: \( \theta_A(\theta_A, \gamma) \) has the following properties:

1) \( \theta_A(\theta_A, \gamma) \) is Lipschitz continuous with respect to \( \theta_A \) and \( \gamma \) when \( \gamma > 0 \).
2) When \( \theta_A \) is stable, \( \lim_{\gamma \to 0} \hat{\theta}_A(\theta_A, \gamma) = \theta_A \).

Proof: i) The Lipschitz continuity of \( P^*(\theta_A, \gamma) \) and \( H^*(\theta_A, \gamma) \) can be proven using Lemma 7 of Appendix A: The second-order growth condition holds because the cost function \( f(\chi, \theta_A, \gamma) \) is a strictly convex quadratic function of \( \chi \) when \( \gamma > 0 \). Consider the difference between \( f(\chi, \theta_A, \gamma) \) and \( f(\chi, \theta_A, \gamma) \), namely \( \| \theta_{a0}^T P - H \|^2_2 - \| \theta_{a0}^T P - H \|^2_2 \). It is Lipschitz continuous with respect to \( P \) and \( H \) modulus \( c\| \theta_A - \theta_A \| \) for some \( c > 0 \) bounded \( P, H \), and \( \theta_A \).

Applying Lemma 7, \( \| \chi^*(\theta_A, \gamma) - \chi^*(\theta_{40}, \gamma) \| \leq c\| \theta_A - \theta_{40} \| \) for some \( c > 0 \). Therefore, the optimal solutions \( P^*(\theta_A, \gamma) \) and \( H^*(\theta_A, \gamma) \) are Lipschitz continuous with respect to \( \theta_A \) for \( \gamma > 0 \). Since \( \hat{\theta}_A(\theta_A, \gamma) = (H^*(\theta_A, \gamma) P^*(\theta_A, \gamma)^{-1})^T \), \( P^*(\theta_A, \gamma) \geq \epsilon_0 I \). \( \hat{\theta}_A(\theta_A, \gamma) \) is Lipschitz continuous with respect to \( \theta_A \) for \( \gamma > 0 \).

Similarly, \( \theta_A(\theta_A, \gamma) \) is Lipschitz continuous with respect to \( \gamma \) with \( \gamma > 0 \).
ii) Let $v^*(\theta_A, \gamma)$ represent the optimal value of the cost function.
When $\gamma = 0$, and $\theta_A$ is stable, the optimal cost value $v^*(\theta_A, 0)$ is 0. Therefore, $\gamma^T P^*(\theta_A, 0) - H^*(\theta_A, 0) = 0$, and $\hat{\theta}_A (\theta_A, 0) = \theta_A$.

From Lemma 6 of Appendix A, $\lim_{\gamma \rightarrow 0} v^*(\theta_A, \gamma) = v^*(\theta_A, 0) = 0$. $v^*(\theta_A, \gamma)$ is a nonnegative quadratic function. Thus,

$$
lim_{\gamma \rightarrow 0} v^*(\theta_A, \gamma) = \lim_{\gamma \rightarrow 0} (||\dot{A}^T P^*(\theta_A, \gamma) - H^*(\theta_A, \gamma)||^2_2 + \gamma ||\mathbf{vec}(P^*(\theta_A, \gamma)), H^*(\theta_A, \gamma)||^2_2) = 0
$$

which implies

$$
\lim_{\gamma \rightarrow 0} (\gamma^T P^*(\theta_A, \gamma) - H^*(\theta_A, \gamma)) = 0,
$$

$$
\lim_{\gamma \rightarrow 0} (\dot{A} - H^*(\theta_A, \gamma)) P^*(\theta_A, \gamma)^{-1} = \lim_{\gamma \rightarrow 0} (\dot{A} - \dot{A}^T (\theta_A, \gamma)) = 0.
$$

Therefore, when $\theta_A$ is stable, $\lim_{\gamma \rightarrow 0} \hat{\theta}_A (\theta_A, \gamma) = \theta_A$. ■

Note from Algorithm 1 that $\hat{\theta}(k)$ is a function of $\theta(k)$. In the following theorem, the boundedness of $\hat{\theta}$ is established from Lemmas 1 and 2. Define the estimation error

$$
\hat{e}(t) = \frac{z(t) - \hat{\theta}(k)^T \phi(t)}{m(t)}, \forall t \in [kT_s, (k + 1)T_s).
$$

(Boundedness of $\hat{e}$ is also established as it is also critical to closed-loop stability of an adaptive control system.

Theorem 3: Algorithm 1 for a parametric model (2) guarantees that

1) $\theta \in l_{\infty}$.
2) $\Delta \hat{\theta}(k)$, where $\Delta \hat{\theta}(k) = \hat{\theta}(k + 1) - \hat{\theta}(k)$.
3) $\hat{e}, \hat{e}_M, \hat{e}_Q \in L_{\infty}$.  
4) If $\theta(k)$ is stable $\forall k > k_c$, where $k_c$ is a finite number, $\lim_{\gamma \rightarrow 0} \hat{\theta}_A = \theta, \lim_{\gamma \rightarrow 0} \hat{e}_M = \hat{e}_M$.

Proof: i) From Lemma 1, $\theta \in l_{\infty}$. From Lemma 5 in Appendix A, $P^*, H^* \in l_{\infty}$ for each fixed $\gamma$. Since $\hat{\theta}_A = (H^* P^* - 1)^T$, $P^* \geq \epsilon I$, we can conclude that $\hat{\theta}_A \in l_{\infty}$, and therefore $\hat{\theta} \in l_{\infty}$.

ii) From Lemma 1, $\Delta \hat{\theta}_2 \in l_{\infty}$. From Lemma 2 (i), $\hat{\theta}_A$ is Lipschitz continuous with respect to $\theta_A$, i.e., $\exists c > 0, \|\Delta \hat{\theta}_A\| \leq c\|\Delta \theta_A\|$. Therefore, $\Delta \hat{\theta}_A \in l_{\infty}$, and $\Delta \hat{\theta}_A \in l_{\infty}$.

iii) $\hat{e}(t) = c(t) + \frac{(\theta(k) - \hat{\theta}(k))^T \phi(t)}{m(t)}, \forall t \in [kT_s, (k + 1)T_s)$.

From Lemma 1, $c \in L_{\infty}$, $\hat{\theta}(k) \in L_{\infty}$, $\theta \in l_{\infty}$, and from Theorem 3 (i), $\theta - \hat{\theta} \in l_{\infty}$. Therefore, $\hat{e} \in L_{\infty}$, Similarly, $\hat{e}_M \in L_{\infty}$

iv) $\hat{e}(t) = c(t) + \frac{(\theta(k) - \hat{\theta}(k))^T \phi(t)}{m(t)}, \forall t \in [kT_s, (k + 1)T_s)$.  

From Lemma 2 (ii), when $\theta_A$ is stable, $\hat{\theta}_A = \theta_A$. Since $\frac{\phi(t)}{m(t)} \in L_{\infty}$, $\lim_{\gamma \rightarrow 0} \hat{e} = \epsilon$. Similarly, $\lim_{\gamma \rightarrow 0} \hat{e}_M = \hat{e}_M$. ■

III. CAIMC FOR AN $n$-TH ORDER SISO PLANT

A. Internal Model Control (IMC) and Its Tracking Error

IMC has an intuitive control structure as shown in Fig. 1, where $G$, $M$, and $Q$ represent the plant, plant model, and plant inverse model, respectively. When the model matches the plant exactly, i.e., $M = G$, the IMC structure becomes open-loop. $Q$ can be designed as an open-loop feedforward controller. For a minimum-phase plant, $Q$ is designed as the inverse of the plant appended with a filter $G_f$ to make it causal, namely $Q = G^{-1} G_f$. The bandwidth of the control system is defined by the bandwidth of $G_f [1]$.

Ideally when $M = G$, we have $\{G_f\} r - y = 0$, where $r$ is the reference signal and $y$ is the plant output. Thus, we consider the tracking error $e = \{G_f\} r - y$, which can be expressed as

$$
e = e_M + e_Q
$$

where

$$
e_M = \{G_f\} (y - y_M)
$$

$$
e_Q = \{G_f\} l - y.
$$

$y_M$ is the model response, and $l = r - y + y_M$ is the input to the approximate inverse $Q$ as shown in Fig. 1. Equation (11) can be derived by noting that $e = \{G_f\} r - y = \{G_f\} (r - y + y_M) + \{G_f\} (y - y_M) - y = \{G_f\} (y - y_M) + \{G_f\} l - y$.

Note that $e_M$ is the filtered difference between the plant and the model responses, and $e_Q$, on the other hand, is the difference between the filtered input to $Q$ and the plant output, which reflects the inverse modeling error.

Using the triangle inequality, $|e| \leq |e_M| + |e_Q|$, the tracking error is upper-bounded by $|e_M| + |e_Q|$. This expression inspires and justifies the separate identifications of $M$ and $Q$. Intuitively, $e_M$ is related to the plant model estimation error $e_M$, and $e_Q$ is related to the inverse estimation error $e_Q$, where $e_M$ and $e_Q$ are the estimation errors defined in (3) for the plant model and inverse parametric model, respectively. Recall that the gradient method (4) is based on minimizing the quadratic cost function of the estimation error $e$. Therefore, the minimization of the quadratic cost function of $e_M$ and $e_Q$ will contribute to reducing the tracking error, $e$. Therefore, we propose CAIMC as shown in Fig. 2, where “composite” means that the model and the inverse are identified simultaneously.

B. Composite Adaptive Internal Model Control

Consider the SISO plant

$$y = \{G\} u
$$

with order $n$ and relative degree $m$. The control objective is that $y$ tracks $\{G\} r$. $G_f$ is an $m$-th-order filter introduced for the causality of the inverse, and it can be designed for the desired bandwidth of the closed-loop system.

The plant is modeled as

$$G = \frac{Z_M}{R_M}
$$

where $Z_M$ is an $(n-m)$-th-order Hurwitz polynomial, and $R_M$ is an $n$-th-order Hurwitz polynomial with leading coefficient 1. The other coefficients of $R_M$ and $Z_M$ are unknown.

The approximate stable and proper inverse is modeled as

$$G^{-1} G_f = \frac{Z_Q}{R_Q}
$$
where $R_Q$ and $Z_Q$ are $n$-th-order Hurwitz polynomials. The leading coefficients of $R_Q^*$ is 1. The other coefficients of $R_Q^*$ and $Z_Q^*$ are unknown.

The following assumptions are made:
A1) Order $n$ of the plant is known.
A2) The relative degree $m$ of the plant is known.
A3) The plant is stable.
A4) The plant is minimum-phase.

Assumptions (A1) and (A2) are standard assumptions for indirect adaptive control. (A3) is a standard assumption for IMC. And (A4) is for the invertibility of the plant. The design procedure of CAIMC for the plant (14), following the certainty equivalence principle, is described in steps as follows:

**CAIMC Design Procedure:**

1. Formulate a plant model and a plant inverse model as (15) and (16).
2. Design $m$-th-order filter $G_f$, whose bandwidth corresponds to the desired closed-loop bandwidth.
3. Derive the parametric models of the proposed plant model and plant inverse model, and identify the unknown parameters using Algorithm 1 in Section II-C2.
4. Treat the identified plant and plant inverse as the true plant and plant inverse, and embed them into the model $M$ and the inverse model $Q$ in the IMC structure, respectively.

### C. Design Detail of CAIMC

This section presents the detail of step 3) and step 4) of the CAIMC design procedure. For step 3), the parametric models for the plant model and plant inverse model are presented. For step 4), the identified parameters are treated as if they were the real ones to implement $M$ and $Q$ in the IMC structure. In the sequel, $Z_M$, $Z_Q$, $R_M$, and $R_Q$ denote the estimates $Z_M^*$, $Z_Q^*$, $R_M^*$, and $R_Q^*$ in (15) and (16), $G_f = \frac{1}{\Lambda_f}$, where $\Lambda_f$ is an $m$-th-order Hurwitz polynomial.

**Plant Model Parameterization and Implementation:**

The goal of the plant model parameterization is to define $z_M$ and $\phi_M$, such that a parametric model $z_M = \theta_M^T \phi_M$ can be used to identify $\theta_M^*$, the unknown parameters of (15). Since a parametric model in the form of (2) for a given physical process is not unique, we attempt to find the particular one such that $\hat{\epsilon}_M m_2^M$ is closely related to $e_M = \{G_f\}(y - y_m)$ as defined in (12), in light of the discussion in Section III-A, so that $e_M$ is minimized in the model identification process.

With simple manipulation of $y = \{Z_Q^*\}u$ according to (15), and introducing a regressor filter $\frac{1}{\Lambda_f}$, we get

$$y = \left\{\frac{\Lambda_M - R_M^*}{\Lambda_M}\right\} u + \left\{\frac{Z_M^*}{\Lambda_M}\right\} u$$

(17)

where $\Lambda_M$ is an $n$-th order-Hurwitz polynomial that serves as a regressor filter. Define

$$z_M = y, \theta_M^* = \{\theta_a^T, \theta_b^T\}^T$$

$$\phi_M = \left[\frac{1}{\Lambda_M} y, \frac{s^{n-1}}{\Lambda_M} y, \frac{1}{\Lambda_M} u, \ldots, \frac{s^{n-m}}{\Lambda_M} u\right]^T$$

(18)

where $\theta_a^*$ is the coefficient vector of $\Lambda_M - R_M^*$ and $\theta_b^*$ is the coefficient vector of $Z_M^*$. Equation (17) can be expressed as $z_M = \theta_M^T \phi_M$. Without considering the stability constraint, the normalized estimation error $\epsilon_m$ is

$$\tilde{e}_M m_2^M = z_M - \theta_M^T \phi_M = y - y_m.$$ 

Recall that from (12), $e_M = \{G_f\}(y - y_m) = \{G_f\}(\hat{\epsilon}_M m_2^M)$. In the following analysis, we show how minimizing $\hat{\epsilon}_M m_2^M$ reduces $e_M$.

With (3) and (10), we can express $\hat{\epsilon}_M m_2^M = \epsilon_M m_2^M + (\theta_M - \hat{\theta}_M)^T \phi_M$. Applying Lemma 1, $\epsilon_M m_2^M \in L_{2r}$. It is later shown in Section IV that all the signals are bounded, so $m_2^M \in L_{\infty}$, which yields $\epsilon_M m_2^M \in L_{2r}$. Since all the signals are bounded, $(\theta_M - \hat{\theta}_M)^T \phi_M \in L_{2r}$. Applying Lemmas 11 and 12, $\|\epsilon_M\|_2 \leq \sup_{\omega \in S_\nu \{G_f\} \omega} \|\hat{\epsilon}_M m_2^M\|_2$. Consequently, the estimation algorithm that minimizes the estimation error $\hat{\epsilon}_M m_2^M$ also reduces $e_M$.

**Plant Inverse Model Parameterization and Implementation:** Similar to the plant identification, the key step in the plant inverse identification is to define a parametric model that directly relates to the inverse modeling error so that $\hat{\epsilon}_Q m_2^Q = e_Q$, where $e_Q = \{G_f\}l - y$ as defined in (13) so that $e_Q$ is minimized in the inverse identification process.

From (14), $\{G_f\}u = \{G^{-1}G_f\}y$, and according to the inverse model (16), $\frac{1}{\Lambda_f}u = \{Z_Q^*\}y$. With a simple manipulation, we can write

$$\frac{1}{\Lambda_f}u = \left\{\frac{\Lambda_Q - R_Q^*}{\Lambda_Q}\right\} \{1\} u + \left\{\frac{Z_Q^*}{\Lambda_Q}\right\} y$$

where $\Lambda_Q$ is an $n$-th-order Hurwitz polynomial that serves as a regressor filter. If we design the inverse parametric model in the same way we designed the plant model parameterization, the associated signals of the parametric model $z_{Q0} = \theta_{Q0}^T \phi_{Q0}$ can
be defined as
\[
\begin{align*}
z_{Q0} &= \left\{ \frac{1}{\Lambda_f} \right\} u, \quad \theta_Q^* = [\theta_c^T, \theta_d^T]^T \\
\phi_Q &= \left[ \left\{ \frac{1}{\Lambda_Q} \right\} y, \ldots, \left\{ \frac{s^n}{\Lambda_Q} \right\} y, \ldots \right] \times \left\{ \frac{1}{\Lambda_Q \Lambda_f} \right\} u, \ldots, \left\{ \frac{s^{n-1}}{\Lambda_Q \Lambda_f} \right\} u \right)^T
\end{align*}
\]
where \( \theta_c^* \) is the coefficient vector of \( Z_0^* \) whose dimension is \( n + 1 \), and \( \theta_d^* \) is the coefficient vector of \( \Lambda_Q - R_Q \) whose dimension is \( n \). If \( Q \) is implemented such that
\[
u = \left\{ (\Lambda_Q - R_Q) \frac{1}{\Lambda_f} \right\} u + \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} l
\]
and then into (21), the estimation error is
\[
z_{Q0} - \hat{\theta}_Q^* \phi_Q_0 = \left\{ \hat{R}_Q \frac{1}{\Lambda_Q \Lambda_f} \right\} u - \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} y + \epsilon_1
\]
\[
= \left\{ \frac{1}{\Lambda_f} \hat{R}_Q \frac{1}{\Lambda_Q} \right\} u - \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} y + \epsilon_1
\]
\[
= \left\{ \frac{1}{\Lambda_f} \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} l - \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} y + \epsilon_1
\]
\[
= \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} \left\{ \left\{ \frac{1}{\Lambda_f} \right\} l - y \right\} + e_2
\]
\[
= \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} e_Q + e_2
\]

where the last three equations are derived from (13) and (22), \( \epsilon_1 \) and \( \epsilon_2 \) are the residual terms resulting from applying the Swapping Lemma in Appendix B. In order to have a more direct and simple relation between \( e_Q \) and \( \hat{e}_Q \), we consider \( \{ X \} \), an operator whose transfer function is \( \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} \). Note that from (23),
\[
e_Q = \{ X \} (z_{Q0} - \hat{\theta}_Q^* \phi_Q_0) - e_2
\]
\[
= \{ X \} z_{Q0} - \{ X \} \hat{\theta}_Q^* \phi_Q_0 - \{ X \} e_2
\]
\[
= \{ X \} z_{Q0} - \hat{\theta}_Q^* \phi_Q = \{ X \} e_2 + \epsilon_3
\]
where \( \epsilon_3 \) is the residual term resulting from applying the Swapping Lemma in Appendix B. If we redefine the inverse parametric model such that the new observation is \( \{ X \} z_{Q0} \) and the new regressor is \( \{ X \} \phi_Q \), then the new estimation error is \( e_Q + \{ X \} e_2 - \epsilon_3 \). Since \( X \) depends on the identified parameters whose values at the current sample time are unavailable, \( z_Q \) and \( \phi_Q \) are defined as
\[
z_Q = \{ X_{k-1} \} u, \quad \theta_Q^* = [\theta_c^T, \theta_d^T]^T
\]
\[
\phi_Q = \{ X_{k-1} \} \left[ \left\{ \frac{1}{\Lambda_Q} \right\} y, \ldots, \left\{ \frac{s^n}{\Lambda_Q} \right\} y, \ldots \right] \times \left\{ \frac{1}{\Lambda_Q \Lambda_f} \right\} u, \ldots, \left\{ \frac{s^{n-1}}{\Lambda_Q \Lambda_f} \right\} u \right)^T
\]
where \( X_{k-1} \) is a transfer function with \( \hat{Z}_Q \) having the parameters identified at the previous sample time (\( k - 1 \)). Without the stability constraints, the normalized gradient algorithm (4) that minimizes \( J(\theta_Q) = \frac{\epsilon_2^2}{m_2^2} \) is adopted to identify \( \theta_Q \), where
\[
\epsilon_2 = \frac{z_{Q0} - \hat{\theta}_Q^* \phi_Q}{m_2}, \quad m_2^2 = 1 + \hat{\theta}_Q^* \phi_Q
\]
are the normalized estimation error and the normalizing term. Algorithm 1 can be used to estimate \( \hat{\theta}_Q \). The inverse model used in CAIMC is still implemented as (22).

To establish the connection between \( e_Q \) and \( \hat{e}_Q \), from (24) and (25), the inverse estimation error can be expressed as
\[
\hat{e}_Q m_2^2 = z_{Q0} - \hat{\theta}_Q^* \phi_Q = (X_{k-1}) \{ z_{Q0} - \hat{\theta}_Q^* \phi_Q \} + e_4
\]
\[
= (X_{k-1}) \{ X \}^{-1} e_Q + e_5 = e_Q + e_6
\]
where \( e_4, e_5, \) and \( e_6 \) are residues from swapping the dynamic operators. These residues are bounded by \( \theta_d(k) - \theta_d(k-1) \). Therefore, for the inverse \( \hat{e}_Q m_2^2 \approx e_Q \) when the parameter adaptation is sufficiently slow.

**D. CAIMC Summary**

The CAIMC scheme is summarized as following:

**Model:**

Plant model: \( \frac{z_Q}{R_M} \). Plant inverse model: \( \frac{z_Q}{R_M} \).

**Parametric Model:**

Model: \( z_M = \theta_Q \phi_M \) with \( z_M \) and \( \phi_M \) defined in (18).
\( \theta_Q = [\theta_c^T, \theta_d^T]^T \), where \( \theta_c^* \) is the coefficient vector of \( \Lambda_M - R_M \) and \( \theta_d^* \) is the coefficient vector of \( Z_M^* \).

Inverse: \( z_Q = \theta_Q^* \phi_Q \) with \( z_Q \) and \( \phi_Q \) defined in (25), \( \theta_Q^* = [\theta_c^T, \theta_d^T]^T \), where \( \theta_c^* \) is the coefficient vector of \( Z_Q^* \), and \( \theta_d^* \) is the coefficient vector of \( \Lambda_Q - R_Q^* \).

**Adaptive Law:**

\( \hat{\theta}_M \) and \( \hat{\theta}_Q \) are identified by Algorithm 1 in Section II-C2. The stability constraints in (8) are imposed on \( \hat{\theta}_a \), \( \hat{\theta}_c \), and \( \hat{\theta}_d \).

**Control Law:**

\( u = \{ (\Lambda_Q - R_Q) \frac{1}{\Lambda_f} \} u + \left\{ \hat{Z}_Q \frac{1}{\Lambda_Q} \right\} l \) as in (22),
where \( l = r - y + y_M, y_M = \theta_M^* \phi_M \) as in (19).

**Remark 1:** Design of the CAIMC algorithm includes the following:

1. \( \hat{G}_f \) is designed based on the desired closed-loop bandwidth.
2. In the parametric model (2) and the unconstrained adaptive law (4), the regressor filter \( \frac{1}{\Lambda} \), the initial condition \( \theta(0) \), and the adaptation gain \( \Gamma \) need to be calibrated. To deal with the unmodeled dynamics, generally, a deadzone is added to the estimation error \( \epsilon \) for robust estimation. The calibration procedure for these parameters is well established [15].
3) In the modified stability-constrained identification (8), $\gamma$ and $\epsilon_0$ need to be chosen. As shown in Lemma 2 (ii), $\gamma$ should be very small for $\hat{\theta}_A$ to be close to $\theta_A$ when $\theta_A$ is stable. $\epsilon_0$ should be small as it is only used for tightening the stability constraint to assure the existence of the optimal solution. One can also choose $\gamma$ to be a sequence $\{\gamma_k\}$ with $\gamma_k \to 0$ as $k \to \infty$. In this case, the rate of convergence for $\gamma_k$ should be carefully chosen as they will affect the parameter convergence of the overall adaptive control system.

IV. STABILITY PROOF OF CAIMC IN THE IDEAL CASE

In this section, stability and asymptotic performance of the ideal $n$-th-order CAIMC is established.

**Theorem 4:** Consider the plant (14) subject to the CAIMC scheme. For any bounded reference $r$, all the signals in the closed-loop system are uniformly bounded. When the regressors are PE, the tracking error $e = \{G_j\}r - y$ converges to 0 as $\gamma \to 0$.

**Proof:** Given that $\Lambda_M$ in (18) and $\Lambda_Q$ in (25) are Hurwitz polynomials with the same order that serve as the regressor filters, we choose $\Lambda_M = \Lambda_Q = \Lambda$ with the coefficient vector $\theta_\lambda \in \mathbb{R}^n$ throughout the proof. Note that the same analysis can be carried out with arbitrary choice of Hurwitz $\Lambda_M$ and $\Lambda_Q$ at the expense of some additional algebra. Defining

$$y_f = \begin{bmatrix} 1 \\ y_f(1) \end{bmatrix}, \quad u_f = \begin{bmatrix} 1 \\ \Lambda y_f \end{bmatrix}, \quad \epsilon_f = \begin{bmatrix} 1 \\ \epsilon_f^2 \end{bmatrix} (\hat{\epsilon}_M m_M^2)$$

we establish signal boundedness in the following steps:

**Step 1.** Correlate $y$ and $u$ to the estimation error: Defining the augmented states $x$ as $y_f, y_f(1), \ldots, y_f(n-1), u_f(1), \ldots, u_f(n+m-1), \epsilon_f(1), \ldots, \epsilon_f(n-1)$, we have

$$\dot{x} = A(t)x + b_1(t)e_M m_M^2 + b_2(t)\tau$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = C(t)x + d_1(t)e_M m_M^2 + d_2(t)\tau$$

where

$$A(t) = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} & 0_{(n-1) \times (n+m)} & 0_{nxm} \\ \theta_{\eta} - \theta_\eta & 0_{nxm} & \theta_{\eta}^T & 0_{nxm} \\ 0_{(n+m) \times (n+m-1)} & I_{n+m-1} & 0_{(n+m-1) \times n} & 0_{n \times n} \\ -\theta_{\eta}^T & 0_{n \times n} & 0_{n \times (n+m)} & -\theta_{\eta}^T \end{bmatrix}$$

$$b_1(t) = \begin{bmatrix} 0_{(n-1) \times 1} \\ 0_{(n+m-1) \times 1} \\ -\theta_{\eta} \\ 0_{n-1} \end{bmatrix}, \quad b_2(t) = \begin{bmatrix} 0_{nx1} \\ 0_{(n+m-1) \times 1} \\ 0_{n-1} \end{bmatrix}$$

$$C(t) = \begin{bmatrix} \hat{\theta}_A^T & \hat{\theta}_{M}^T & 0_{1 \times m} & 0_{1 \times n} \\ 0_{1 \times n} & \theta_{\eta}^T & 0_{1 \times n} & 0_{1 \times n} \\ \hat{\theta}_{M}^T - \hat{\theta}_{M}^T \eta \theta_t - \hat{\theta}_{M}^T \end{bmatrix}$$

$$d_1(t) = \begin{bmatrix} 0 \\ -\eta \\ 0 \end{bmatrix}, \quad d_2(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\hat{\theta}_a = \hat{\theta}_c, \hat{\theta}_\epsilon \in \mathbb{R}^n$ and $\eta$ is the $(n+1)$-th entry of $\hat{\theta}_c, \tau = \{ZQ\gamma\}r, \hat{\theta}_{M}^T \in \mathbb{R}^{n+m}$, and $\hat{\theta}_{M}^T \in \mathbb{R}^{n+m}$ are the coefficient vectors of $\hat{Z}_M \Lambda_f, \hat{R}_Q \Lambda_f$, and $\Lambda_{\epsilon}$, respectively. The derivations of (28) are given in Appendix C.

**Step 2.** Establish exponential stability of the homogeneous part of (28): $A(t)$ has a block upper triangular structure, whose eigenvalues are the same as the eigenvalues of its diagonal matrices, i.e., for each fixed time $t$, $A(t)$ has the same eigenvalues as $A_1 = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \theta_{\eta} - \theta_\eta & \theta_{\eta}^T - \theta_\eta^T \end{bmatrix}$, $A_2 = \begin{bmatrix} 0_{(n+m-1) \times 1} & I_{m-1} \\ \theta_{\eta} - \theta_\eta & \theta_{\eta}^T - \theta_\eta^T \end{bmatrix}$, and $A_3 = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ \eta \theta_{\eta} - \hat{\theta}_{M}^T \end{bmatrix}$.

Since $\hat{\theta}_a$ is the coefficient vector of $\Lambda = \hat{R}_M$, the eigenvalues of $A_1$ are the solutions of $\hat{R}_M \hat{R}_M$, which have negative real parts $\forall \tau \geq 0$ because of the stability constraint enforced in the parameter identification, as discussed in Section II. $\theta_{\eta}^T$ is the coefficient vector of $\hat{R}_Q \Lambda_f$, so the eigenvalues of $A_2$ are the solutions of $\hat{R}_Q \Lambda_f = 0$, which also have negative real parts $\forall \tau \geq 0$ using the same argument. The eigenvalues $A_3$ are the solutions of $\Lambda = 0$ which also have negative real parts.

By Theorem 3, the constrained identification guarantees that $\hat{\theta}_a, \hat{\theta}_c, \hat{\theta}_\epsilon, \hat{\theta}_d \in \mathbb{R}^n$, $\Delta \theta_\epsilon, \Delta \theta_\epsilon, \Delta \theta_\epsilon, \Delta \theta_d \in \mathbb{R}$. Their zero-order hold (ZOH) signals are used in $A(t)$. Thus, $A(t)$ is piece-wise differentiable with respect to $t$. $\|A(t)\| \in \mathcal{L}_\infty$. Applying Lemma 9 of Appendix B, let $k_0 = T_\epsilon$, the system is exponentially stable, and the state transition matrix $\Phi(\tau, t)$ associated with $A(t)$ satisfies $\|\Phi(\tau, t)\| \leq k_1 e^{-k_2 (\tau - t)}, \forall t \geq \tau \geq 0$ for some constants $k_1, k_2 > 0$.

**Step 3.** Establish signal boundedness: The $\mathcal{L}_\infty$ norm $\|\bullet\|_{28}$ for some $\delta > 0$ is the exponentially weighted $c_\delta$ norm defined as $\|x(t)\|_{28} := (\int_0^\infty e^{-\delta (\tau - t)} \tau x(t) d\tau)^{\frac{1}{2}}$. Applying Lemma 10 of Appendix B to the state-space equation (28), we can obtain

$$\|x(t)\|_{28} \leq c \|\hat{\epsilon}_M m_M^2\|_{\tau} + c$$

$$\|x(t)\|_{28} \leq c \|\hat{\epsilon}_M m_M^2\|_{\tau} + c$$

where $\|\bullet\|$ is a vector norm, for any $\delta \in [0, \delta_1]$ where $\delta_1 > 0$ is any constant less than $2k_2$ and some finite constant $c \geq 0$. 

Authorized licensed use limited to: University of Michigan Library. Downloaded on August 30,2020 at 03:05:21 UTC from IEEE Xplore. Restrictions apply.
For simplicity of the representation in this paper, \( c \) is used to represent a generic constant.

We define the fictitious normalizing signal \( m_f^2 := 1 + \| u_1 \|_2^2 + \| y \|_2^2 \). From the state-space equation, we have \( \| u_1 \|_2 + \| y \|_2 \leq c \| x \|_2 + c \| (\Theta M) m_f \|_2 + c \). With (29), we have \( \| u_1 \|_2 + \| y \|_2 \leq c \| (\Theta M) m_f \|_2 + c \), implying

\[
m_f^2 \leq c \| (\Theta M) m_f \|_2 + c.
\]

From (18), applying Lemma 13 of Appendix B,

\[
m_M := \sqrt{1 + \phi^T \Theta c} \leq c \phi \hspace{1cm} m_f^2 \leq c \| \Theta \phi \|_2 + c \tag{30}
\]

where \( \bar{y} = \hat{\Theta} M \phi \in \mathcal{L}_2 \), or

\[
m_f^2 \leq c \int_0^1 e^{-\delta (t-\tau)^2} \phi^T \Theta \phi \, d\tau + c
\]

where \( 0 < \delta \leq \delta^* = \min \{2\delta, \delta_1, \} \), \( \delta_1 \in (0,2k_2) \).

Applying Bellman–Gronwall–Wallma [15], we can establish that \( m_f \in \mathcal{L}\). Then with (30), we have \( m_M \in \mathcal{L}_\infty \) and therefore \( \phi_M, x, \hat{x}, u, \{ G_f \} y \in \mathcal{L}_\infty \).

First, from the plant model estimation error \( \hat{\Theta} M m_f^2 = y - y_M \) in (20). With the assumption that \( \phi_M \) is PE, \( \theta_M \to \theta_M^* \) exponentially according to Lemma 1. Therefore, \( \hat{\Theta} M m_f^2 \to \hat{\Theta} M \phi_M = (\theta_M - \theta_M^*)^T \phi_M \to 0 \) as \( t \to \infty \), and \( \theta_M \) satisfies the stability constraints in (8). Then establish convergence of \( e \) by demonstrating convergence of \( e_M \) and \( e_Q \).

Next, with the persistent excitation condition, \( e_M \) and \( e_Q \) can be expressed as the sum of \( \Theta \phi \) and \( e_Q \), respectively.

For simplicity of the representation in this paper, \( e \) is used to represent a generic constant.

We define the fictitious normalizing signal \( m_f^2 := 1 + \| u_1 \|_2^2 + \| y \|_2^2 \). From the state-space equation, we have \( \| u_1 \|_2 + \| y \|_2 \leq c \| x \|_2 + c \| (\Theta M) m_f \|_2 + c \). With (29), we have \( \| u_1 \|_2 + \| y \|_2 \leq c \| (\Theta M) m_f \|_2 + c \), implying

\[
m_f^2 \leq c \| (\Theta M) m_f \|_2 + c.
\]

From (18), applying Lemma 13 of Appendix B,

\[
m_M := \sqrt{1 + \phi^T \Theta c} \leq c \phi \hspace{1cm} m_f^2 \leq c \| \Theta \phi \|_2 + c \tag{30}
\]

where \( \bar{y} = \hat{\Theta} M \phi \in \mathcal{L}_2 \), or

\[
m_f^2 \leq c \int_0^1 e^{-\delta (t-\tau)^2} \phi^T \Theta \phi \, d\tau + c
\]

where \( 0 < \delta \leq \delta^* = \min \{2\delta, \delta_1, \} \), \( \delta_1 \in (0,2k_2) \).

Applying Bellman–Gronwall–Wallma [15], we can establish that \( m_f \in \mathcal{L}\). Then with (30), we have \( m_M \in \mathcal{L}_\infty \) and therefore \( \phi_M, x, \hat{x}, u, \{ G_f \} y \in \mathcal{L}_\infty \).

First, from the plant model estimation error \( \hat{\Theta} M m_f^2 = y - y_M \) in (20). With the assumption that \( \phi_M \) is PE, \( \theta_M \to \theta_M^* \) exponentially according to Lemma 1. Therefore, \( \hat{\Theta} M m_f^2 \to \hat{\Theta} M \phi_M = (\theta_M - \theta_M^*)^T \phi_M \to 0 \) as \( t \to \infty \), and \( \theta_M \) satisfies the stability constraints in (8). Then establish convergence of \( e \) by demonstrating convergence of \( e_M \) and \( e_Q \).

Next, with the persistent excitation condition, \( e \) can be expressed as the sum of \( \Theta \phi \) and \( e_Q \), respectively.

Finally, the tracking performance improves as the identified parameters approach \( \theta \), which is stable.

Remark 2: Note that \( \hat{\Theta} M \) does not appear in the closed-loop representation (28); therefore, the property of \( \hat{\Theta} M \) is not required for establishing stability of CAIMC. It is only needed for establishing convergence of the tracking error.

Remark 3: The regularizor term in the modified stability-constrained identification is critical for the closed-loop stability analysis of CAIMC. As shown in step 2 of the proof, boundedness and continuity of the modified stability-constrained identification is crucial for establishing exponential stability of the homogeneous part of the closed-loop state-space equation using Lemma 9. The modified stability-constrained identification is general in the sense that it can be applied to any adaptive scheme where boundedness and continuity of the signals are required for establishing closed-loop stability.

Remark 4: Theorem 4 shows that the tracking error \( e \to 0 \) as \( \gamma \to 0 \) and \( t \to \infty \) with the persistent excitation condition. \( \gamma \), however, has to be nonzero to assure that the optimization problem (8) has an unique optimal solution. Therefore, the implication when \( \gamma \) is a small nonzero number is discussed.

According to Lemma 2, when \( \theta \) satisfies the stability constraints in (8), \( \hat{\Theta} M \phi_M \) is Lipschitz continuous with respect to \( \gamma \), \( m_M \to 0 \), \( m_Q \to 0 \), \( m_f \to 0 \), and \( e_M \to 0 \). Then according to (11), we have \( e_M \to 0 \). With persistent excitation, \( \hat{\Theta} M m_f^2 \), \( e_M m_Q^2 \), and \( e_6 \) all converge to zero exponentially according to Lemma 1. Exponential convergence of the tracking error to zero is guaranteed.

Remark 5: For simplicity and clarity, the stability analysis is performed under the assumption that there are no unmodeled dynamics. In general, however, there are unmodeled dynamics in the presentation of the physical plant. To handle the unmodeled dynamics, a deadzone is typically added to the estimation error for robustness [15]. The robust CAIMC stability proof follows a similar procedure by expanding the proof here as shown in [15] for robust adaptive pole placement control, and its tracking error \( e \) is bounded.

V. CAIMC SIMULATION RESULTS

In this section, the CAIMC scheme is applied to the third-order LTI plant (9). Algorithm 1 is used to identify the unknown parameters of the plant and its inverse simultaneously. With this example, we demonstrate that the unconstrained adaptive law yields unstable estimates due to transients even when the initial estimates and the true parameters are stable, and Algorithm 1 guarantees stability of the estimated parameters. The tracking performance improves as the identified parameters converge.

Equation (9) is stable and minimum-phase, with relative degree 1. Assume that the desired bandwidth is approximately 0.8 Hz, then a first-order filter \( G_f = \frac{1}{\theta_d + 1} \) is adopted.

The plant inverse modeled from the inverse of (9) appended with the filter \( G_f \) is

\[
u = \left\{ \begin{array}{l}
\theta^*_1 s^3 + \theta^*_2 s^2 + \theta^*_3 s + \theta^*_4 d \\hspace{1cm} s^3 + \theta^*_2 d^2 s^2 + \theta^*_1 d^2 s + \theta^*_0 d
\end{array} \right\}
\]

where \( \theta^*_1 = [2.5, 5.5, 7.5]^T \), \( \theta^*_2 = [6, 5.5, 2.5]^T \) are assumed to be unknown. The inverse is stable. There are no unmodeled dynamics.

The CAIMC scheme, as summarized in Section III, is applied to the plant. The plant (9) and its inverse (31) are identi-
CAIMC simulation result.

\[ \hat{f} \chi \] is the objective or cost function. The optimal value \( \Phi \) such that \( \Phi \) is Lipschitz continuous with respect to \( \chi \) modulo \( \kappa \) on \( \Phi \cap N \), i.e., \( \exists \kappa < \infty, \| (f(\chi_1, p) - f(\chi_2, p_0)) - (f(\chi_2, p) - f(\chi_1, p_0)) \| \leq \kappa \| \chi_1 - \chi_2 \|, \forall \chi_1, \chi_2 \in \Phi \cap N. \)

Then, \( \| \hat{\chi}(p) - \hat{\chi}^*(p_0) \| \leq c^{-1} \kappa. \)

**Appendix B**

**Mathematical Preliminaries for Stability Proof**

The stability proof of CAIMC is done by representing the closed-loop system as a linear time-varying (LTV) system. Rele-
vant results are introduced here to establish exponential stability and signal boundedness of linear systems.

Lemma 8. Swapping Lemma: [15] Let \( \hat{\theta} \) be differentiable, and \( \omega : \mathbb{R}^+ \rightarrow \mathbb{R}^n \). Let \( W \) be a proper stable rational transfer function with a minimal realization \((A_W, B_W, C_W, d_W)\). Then,

\[
\{W\} \hat{\theta}^T \omega = \hat{\theta}^T \{W\} \omega + \{W\}_{\epsilon}(\{W\}_{\epsilon}^T \hat{\theta})
\]

where \( W_c = -C_W^T (sI - A_W)^{-1} \), and \( W_b = (sI - A_W)^{-1} B_W \).

Lemma 9: Consider an LTV system \( x = A(t) x \), where \( x \in \mathbb{R}^n \), and the elements of \( A(t) \) are piecewise differentiable and bounded. Assume that \( \text{Re}\{\lambda_i(A(t))\} < -\delta_0 \) for all \( t \geq 0 \) and for \( i = 1, 2, \ldots, n \), where \( \delta_0 > 0 \) is some constant. Also, assume that \( \|A(t)\| \leq c \), for some constant \( c > 0 \), \( \forall t \geq 0 \), where \( \|A(t)\| \) is the induced norm.

If \( \exists \delta_0 > 0, \sup_{0 \leq \tau \leq \delta_0} \|A(t + \tau) - A(t)\| \leq \delta_0 \), then the equilibrium state \( x_e = 0 \) is exponentially stable, i.e., the state transition matrix

\[
\|F(t, \tau)\| \leq \lambda_0 e^{-\alpha_0(t - \tau)}, \forall t \geq \tau \geq 0
\]

for some \( \lambda_0, \alpha_0 > 0 \) [31].

Lemma 10: Consider an LTV system \( x = A(t) x + B(t) u \), where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^r \), \( u \in \mathbb{R}^m \), and the elements of the matrices \( A(t), B(t) \) are bounded piecewise continuous functions of time. If the state transition matrix \( \|F(t, \tau)\| \leq \lambda_0 e^{-\alpha_0(t - \tau)} \) for some \( \lambda_0, \alpha_0 > 0 \) and \( u \in L_2 \), then for any \( \delta \in [0, \delta_0] \) where \( 0 < \delta_0 < 2 \alpha_0 \) is arbitrary, we have

\[
\begin{align*}
\|x(t)\| & \leq \frac{\delta_0}{\alpha_0} \|u\|_{\delta_0} + \epsilon_1 \\
\|x(t)\|_{\delta_2} & \leq \frac{\delta_0}{\sqrt{1 - \delta_0^2}} \|u\|_{\delta_2} + \epsilon_2
\end{align*}
\]

where \( \epsilon_1 = \sup_{0 \leq \tau \leq \delta_0} \|B(t)\| \) and \( \epsilon_2 \) is an exponentially decaying to zero term due to the initial condition [15].

Lemma 11: Consider an LTI system \( y = \{H\} u \). If \( u \in L_2 \) and \( h \) is the impulse response of \( H \), then

\[
\|y\|_2 \leq \sup_{\omega} \|H(j\omega)\| \|u\|_2.
\]

Lemma 12: Let \( H \) be a strictly proper rational function of \( s \). Then, \( H \) is analytic in \( \text{Re}[s] \geq 0 \) if and only if \( h \in L_1 \).

Lemma 13: Consider an LTI system \( y = \{H\} u \), where \( H \) is strictly proper and analytic in \( \text{Re}[s] \geq -\frac{\alpha}{2} \) for some \( \alpha > 0 \) and \( u \in L_2 \). Then, we have \( \|y(t)\| \leq c \|u\|_{\delta_2} \) for some \( c > 0 \).

Lemma 14. Bellman–Gronwall (B-G) Lemma: [15] Let \( \lambda(t), g(t), k(t) \) be nonnegative piecewise continuous functions of time \( t \). If a function \( f(t) \) satisfies the inequality \( f(t) \leq g(t) \int_{t_0}^t k(s) f(s) ds + \lambda(t) \), \( \forall t \geq t_0 \geq 0 \), then

\[
f(t) \leq g(t) \int_{t_0}^t \lambda(s) k(s) \left[ \exp \left( \int_s^t k(\tau) g(\tau) d\tau \right) \right] ds + \lambda(t)
\]

\( \forall t \geq t_0 \geq 0 \).

In particular, if \( \lambda(t) \equiv \lambda \) is a constant and \( g(t) \equiv 1 \), then

\[
f(t) \leq \lambda \exp \left( \int_{t_0}^t k(s) ds \right), \forall t \geq t_0 \geq 0.
\]

APPENDIX C

DERIVATION OF THE CLOSED-LOOP STATE-SPACE EQUATION OF CAIMC

Let \( S_n \) represent the vector \([1, s, \ldots, s^{n-1}]^T\). Combining (18), (19), and (27), let \( \hat{\theta}_c \in \mathbb{R}^{n+m} \) be the coefficient vector of \( \hat{Z}_M \Lambda_f \), we have \( \hat{\theta}_c \hat{m}_M \|^2 = z_M - \hat{\theta}_c^T \phi_M = \{s^n\} y_f + \theta_n^T \{S_n\} y_f - \theta_t^T \{S_n\} u_f - \hat{\theta}_c^T \{S_n\} u_f \). Therefore,

\[
y_f(t^n) = (\hat{\theta}_c - \theta_c)^T \{S_n\} y_f + \hat{\theta}_c^T \{S_n\} u_f + \hat{\theta}_c^T \{S_n\} u_f
\]

From (22) and (27), and \( l = r - \hat{\theta}_c^T \{S_n\} u_f \), let \( \tau = \{Z_Q \frac{1}{\lambda}\} r \) and \( \hat{\theta}_c \hat{m}_M \|^2 \)
which \( \hat{\theta}_c \), \( \eta \) is the \((n+1)^{th}\) entry of \( \hat{\theta}_c \). Then,

\[
\{\hat{R}_Q \Lambda_f\} u_f = \tau - \{\hat{Z}_Q \frac{1}{\lambda}\} \hat{m}_M \|^2
\]

\[
= \tau - \hat{\theta}_c^T \{S_n\} \hat{\theta}_c - \eta \hat{\theta}_c^T \{S_n\} \hat{\theta}_c - \eta \hat{\theta}_c^T \{S_n\} \hat{\theta}_c \]

\[
= \tau - (\hat{\theta}_c^T \{S_n\} \hat{\theta}_c - \eta \hat{\theta}_c^T \{S_n\} \hat{\theta}_c - \eta \hat{\theta}_c^T \{S_n\} \hat{\theta}_c)
\]

Then

\[
\hat{\theta}_Q \in \mathbb{R}^{n+m} \]

represent the coefficient vector of \( \hat{R}_Q \Lambda_f \).

Then

\[
u_f^{(n+m)} = -\hat{\theta}_Q^T \{S_n+m\} u_f - (\hat{\theta}_c - \eta \hat{\theta}_c)^T \{S_n\} \hat{\theta}_c - \eta \hat{\theta}_c^T \{S_n\} \hat{\theta}_c
\]

\[
= \tau - \hat{\theta}_c^T \{S_n\} \hat{\theta}_c + \hat{\theta}_c^T \{S_n\} \hat{\theta}_c \]

Combining (34), (35), and (36), we have the state-space equation as shown in (28).

REFERENCES


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