WEAK ASYMPTOTIC CAUSALITY AND THE HAAG-RUELLE S MATRIX: Summary of Results

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ABSTRACT: We prove the Stapp-Chandler weak asymptotic causality condition for the Haag-Ruelle S matrix in the regimes of rapid decrease (smooth wavepackets) and exponential decrease (shrinking Gaussian, analytic wave packets).

I. QUASI-WAVEPACKETS: RUELLE LOCALIZATION

Let $f(x,t) = \int d^4p \, e^{i(p^0-\omega)t} \, e^{-ip\cdot x} \, \Phi(p)$

$$\Phi(p) \equiv h(p^0\omega) \, \varphi(\underline{p}) \in \mathcal{D}(R^4),$$

where h isolates the one-particle spectrum (one type of massive, scalar particle). Translation by a four-vector is defined to act on the mass shell restriction:

$$f^a(x,t) = \int d^4p \, e^{i(p^0-\omega)t} e^{-ip\cdot x} \, e^{i\hat{p}\cdot a} \, \Phi(p)$$

$$\hat{p} \equiv (\omega, \underline{p}).$$

The quasi-wavepacket $f(x,t)$ has its essential support at $t = x^0$ and in the velocity cone (K. Hepp)

$$\mathcal{C}(\mathrm{supp}\,\varphi) = \{(t,\underline{x}): \underline{x} = (\underline{p}/\omega)t \text{ with } \underline{p} \in \mathrm{supp}\,\varphi\}.$$

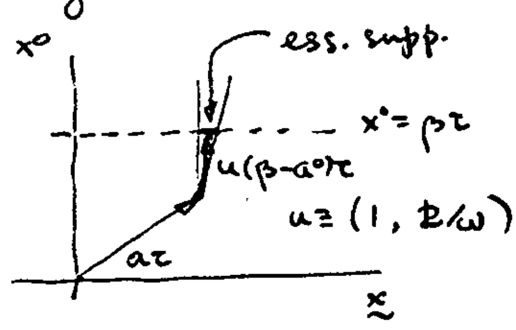
II. UNIFORM SPACELIKE SEPARATION OF FUTURE (PAST) VELOCITY CONES

Consider the quasi-local states

$$|\Phi_1^{a_1\tau}(\beta_1\tau)\cdots\Phi_n^{a_n\tau}(\beta_n\tau)\rangle \equiv \prod_{i=1}^{n} A(f_i^{a_i\tau},\beta_i\tau)^*\,\Omega$$

where $A(f,t) = \int d^4x\, f(x,t)\, A(x)$. The field operator $A(x)$ is assumed to be quasilocal according to the Wightman-Hepp axioms, and to create one-particle states from the vacuum.

Each field operator in the above product has essential support in x-space in the intersection of the plane $x^0 = \beta\tau$ with the velocity cone of φ, but with vertex shifted to a, as shown in the diagram below. That is the set of points $x = a\tau + u(\beta - a^0)\tau$, where $u = (1, \underset{\sim}{k}/\omega)$ is one of the velocities in supp φ.



We say that the set of translations $(a_1,\ldots,a_n)$ <u>uniformly</u>, <u>spacelike</u> <u>separates</u> the velocity cones of the mass shell wave packets φ_i in the <u>future</u> if the following construction is valid:

(i) There is a set of times $(\beta_1,\ldots,\beta_n)$ with $\beta_i - a_i^0 > 0$, such that

(ii) there is another set of times $(T_1, \ldots, T_n)$ with $\beta_i - T_i > 0$ such that, if we define

$$x_i(\kappa) = u_i(\beta_i - T_i) + \kappa[u_i(T_i - a_i^0) + a_i]$$

$$0 \leq \kappa \leq 1,$$

then for $i \neq j$, $x_i(\kappa) - x_j(\kappa)$ is spacelike and uniformly bounded away from the lightcone for $0 \leq \kappa \leq 1$ and for all $u_i = (1, \underline{p}_i/\omega_i)$ in some closed neighborhood of supp φ_i (same for u_j).

The geometrical content of this condition is roughly the following: there exists a spacelike surface such that the velocity cones with vertices at a_i intersect, if at all, behind it.

We define the uniform spacelike separation of the translated velocity cones in the past in the analogous way.

III. UNIFORM RAPID CONVERGENCE OF TRANSLATED HAAG-RUELLE STATES

Lemma: Consider the translated Haag-Ruelle state $|\Phi_1^{a_1 T}(\beta_1 T) \ldots \Phi_n^{a_n T}(\beta_n T)\rangle$. Let the translations uniformly spacelike separate the velocity cones in the $\binom{\text{future}}{\text{past}}$ relative to the times β_i. Let

$$\operatorname{sgn} \tau = \operatorname{sgn}\begin{pmatrix}\text{future}\\ \text{past}\end{pmatrix} = \operatorname{sgn}(ex), \text{ where } ex = \begin{pmatrix}\text{out}\\ \text{in}\end{pmatrix}.$$

Then $\| \, |\Phi_1^{a_1\tau}(\beta_1\tau) \cdots \Phi_n^{a_n\tau}(\beta_n\tau)\rangle - |\varphi_1^{a_1\tau} \cdots \varphi_n^{a_n\tau} \, ex\rangle \| = o(\tau^{-\infty})$.

This theorem is a result of K. Hepp, reorganized a bit. We did part of the arrangement during a collaboration with J. Bros.

IV. QUASI-WAVEPACKETS: EXPONENTIAL LOCALIZATION

Now let

$$\Phi_{\xi,\eta}(p) = \Phi(p) \exp - [(p^0-\omega)^2 \xi + (\underset{\sim}{p} - \underset{\sim}{P})^2 \eta].$$

Let $h(p^0-\omega)$ be analytic in a disc of radius μ and have support in $(\mu+\varepsilon, -\mu-\varepsilon)$; and let $\varphi(\underset{\sim}{p})$ be analytic near $\underset{\sim}{p} = \underset{\sim}{P}$, still in $\mathcal{D}(\mathbb{R}^3)$.

The "shrinking Gaussian" mass shell restriction in x-space

$$\hat{f}_{\eta|\tau|}(u\tau) = \int d^3p \; e^{-i\hat{p}\cdot u\tau} \; e^{-(\underset{\sim}{p}-\underset{\sim}{P})^2 \eta|\tau|} \; \varphi(\underset{\sim}{p})$$

decreases exponentially for all directions u not along $(\sqrt{\underset{\sim}{P}^2+m^2}, \underset{\sim}{P})$. [Omnès, Iagolnitzer, Stapp].

The quasi-wavepacket $f_{\frac{\beta|\tau|}{\sqrt{2}\mu}, \eta|\tau|}(x, p\tau)$ has that property, plus sufficiently uniform exponential decrease in $|x^0-t|$ to give the next lemma.

V. EXPONENTIAL CONVERGENCE OF TRANSLATED, SHRINKING GAUSSIAN, HAAG-RUELLE STATES

<u>Lemma</u>: Let the geometrical conditions be the same as before. Let the quasi-wavepackets be analytic shrinking Gaussians of the sort just described. Let the field be exponentially quasi-local. Then there are finite constants C and α independent of τ such that

$$\left\| \,|\Phi_{1\,\frac{\beta|\tau|}{2\mu_1},\,\eta|\tau|}^{a_1\tau}(\beta_1\tau)\cdots\rangle - |\varphi_{1\,\eta|\tau|}^{a_1\tau}\cdots\varphi_{n\,\eta|\tau|}^{a_n\tau}\ ex\rangle \right\|$$

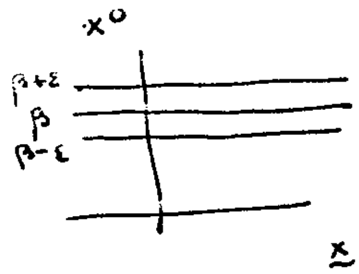
$$\leq C \exp - \alpha|\tau| ,$$

where $\operatorname{sgn}\tau = \operatorname{sgn} ex$.

VI. ACAUSAL TRANSLATION OF IN AND OUT STATES (Chandler and Stapp)

We modify their definition only slightly. Consider the scattering amplitude $\langle\psi_1\cdots\psi_m\ \text{out}\,|\,\varphi_1\cdots\varphi_n\ \text{in}\rangle$ with p-space wave-packets in $\mathcal{D}$. The translation $\langle\psi_1^{b_1\tau}\cdots\psi_m^{b_m\tau}\ \text{out}\,|\,\varphi_1^{a_1\tau}\cdots\varphi_n^{a_n\tau}\ \text{in}\rangle$ is said to be <u>acausal</u> if there is a spatial plane $\beta = x^0$ such that $(a_1,\ldots,a_n)$ uniformly spacelike separates the velocity cones of the incoming particles in its past and $(b_1,\ldots,b_m)$ uniformly spacelike separates the velocity cones of the outgoing particles in its future.

Geometrically, there is a time-slice centered at $\beta = x^0$ such that all translated incoming velocity



cones have their vertices and intersect, if at all in the future of $\beta+\varepsilon$ and all outgoing cones have their vertices and intersect only in the past of $\beta-\varepsilon$.

As the dilation parameter τ goes to plus infinity, the incoming particles can "interact" only far in the future of the space-time regions where the outgoing particles "emerge", which violates elementary causality.

VII. WEAK ASYMPTOTIC CAUSALITY

Theorem: For an acausal translation of (1) Ruelle localized wavepackets and the HR Smatrix corresponding to quasilocal fields, or (2) analytic, shrinking Gaussian wavepackets and the HR Smatrix corresponding to exponentially quasilocal fields: $(0 \le \tau < \infty)$

(1) $|\langle \psi_1^{b_1\tau} \cdots \psi_m^{b_m\tau}\ \mathrm{out} | \varphi_1^{a_1\tau} \cdots \varphi_n^{a_n\tau}\ \mathrm{in}\rangle| = o(\tau^{-\infty})$;

(2) $|\langle \psi_{1\,\eta_1|\tau|}^{b_1\tau} \cdots \psi_{m\,\eta_m|\tau|}^{b_m\tau}\ \mathrm{out} | \varphi_{1\,\lambda_1|\tau|}^{a_1\tau} \cdots \varphi_{n\,\lambda_n|\tau|}^{a_n\tau}\ \mathrm{in}\rangle| = O(\exp - \gamma\tau)$.

VIII. CLUSTER PROPERTIES

By putting more detailed restrictions on the widths of the velocity cones, we can allow the translated vertices of the cones in an acausal translation to be on <u>either</u> side of the spatial plane $x^0 = \beta$. The theorem still holds, and then it includes a certain class of timelike cluster laws (a rather small one) where the time may go to either plus or minus infinity.

The lemma in Sec. V permits the extension of Hepp's results on the rapid approach to the limit in the spatial cluster laws for disjoint velocities, where the impact parameters are increasing, to the exponential regime, if the wave packets are analytic, shrinking Gaussians.

A similar remark applies to his results on the vacuum structure of the S matrix, which corresponds to another class of timelike cluster laws.