One-Photon States and Lorentz Invariance*

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Contents

1 Introduction

This is a LaTeX version of a manuscript containing more systematic notes for some lectures from a graduate course I gave in Fall, 1973. It provides a conceptual description of Lorentz/Poincaré covariance for photons, from the point of view of the Lorentz gauge class, and the radiation gauge within that class.

Our convention for the Lorentz metric is $(+---)$, with four-vector indices, $\mu, \nu, \dots$, taking values $0, 1, 2, 3$, and three-vector indices, $i, j, \dots$, taking values $1, 2, 3$.

2 Lorentz gauge wave functions

Let $\mathcal{H}_{\mathrm{L}}$ be the linear space of four-vector valued, Lebesgue-measureable functions of $\mathbf{k} \in \mathbb{R}^3$, subject to the Lorentz condition,

$$k \cdot f(\mathbf{k}) = \omega f^0 - \mathbf{k} \cdot \mathbf{f} = 0\,, \tag{2.1a}$$

$$k = (\omega, \mathbf{k}), \qquad \omega \equiv |\mathbf{k}| \geq 0\,, \tag{2.1b}$$

and endowed with the pseudo inner product

$$(f,\, f) = -\int \frac{\mathrm{d}^3 k}{2\omega}\, \bar{f} \cdot f. \tag{2.2}$$

The inner product is nonnegative, because the Lorentz condition implies that the real and imaginary parts of the complex four-vector f are lightlike or spacelike. There are, however, nonzero vectors of zero pseudonorm, namely, those of the form $k^\mu h$, where h is a complex scalar function.

To check that all zero-length vectors have this form, it is convenient to introduce a basis of four, independent four-vectors,

$$k\,, \quad \widetilde{k} \equiv (\omega, -\mathbf{k})\,, \quad e_\lambda(k), \quad \lambda = 1, 2\,, \tag{2.3}$$

where the spacelike vectors e_λ are chosen so that

$$\begin{aligned} k \cdot e_\lambda &= \widetilde{k} \cdot e_\lambda = 0 \quad \Rightarrow \quad \mathbf{e}_\lambda \cdot \mathbf{k} = 0\,, \\ e_\lambda \cdot e_{\lambda'} &= -\delta_{\lambda\lambda'}\,, \\ e_\lambda &= \left(0, \mathbf{e}_\lambda\right). \end{aligned} \tag{2.4}$$

Note that $k \cdot k = \widetilde{k} \cdot \widetilde{k} = 0$, and $k \cdot \widetilde{k} = 2\omega^2$. That we can always choose such a basis follows by looking in the special Lorentz frame

$$k = (\omega, 0, 0, \omega), \qquad \widetilde{k} = (\omega, 0, 0, -\omega). \tag{2.5}$$

Then we can choose, for example,

$$e_1 = (0, 1, 0, 0), \qquad e_2 = (0, 0, 1, 0). \tag{2.6}$$

Now any $f(\mathbf{k})$ can be written

$$f^\mu(\mathbf{k}) = k^\mu f_\parallel(\mathbf{k}) + \tilde{k}^\mu f_{\tilde{\parallel}}(\mathbf{k}) + \textstyle\sum_\lambda e^\mu_\lambda f_\lambda(\mathbf{k}). \tag{2.7}$$

If $f \in \mathcal{H}_{\rm L}$, then the Lorentz condition implies that $f_{\tilde{\parallel}} = 0$. If f is a zero length vector in $\mathcal{H}_{\rm L}$, then

$$\begin{aligned} 0 = (f, f) &= \textstyle\sum_\lambda \displaystyle\int \frac{\mathrm{d}^3 k}{2\omega} \left| f_\lambda(\mathbf{k}) \right|^2 \\ \Rightarrow \quad f_\lambda(\mathbf{k}) = 0 \quad &\Rightarrow \quad f = k f_\parallel \,. \end{aligned} \tag{2.8}$$

Moreover, the representation implies that any zero-length vector in $\mathcal{H}_{\rm L}$ is orthogonal to *all* vectors in $\mathcal{H}_{\rm L}$:

$$(f, f) = 0 \quad \Rightarrow \quad (f, g) = 0 \quad \text{for all} \quad g \in \mathcal{H}_{\rm L} \,. \tag{2.9}$$

The set $\mathcal{R}$ of zero-length vectors is called the *radical* of $\mathcal{H}_{\rm L}$.[1] The space $\mathcal{H}_{\rm L}$ with its pseudo inner product is a pseudo Hilbert space, because it has a radical. We define the physical Hilbert space $\mathcal{H}$ of one-photon states by "dividing out the radical":

$$\mathcal{H} \equiv \mathcal{H}_{\rm L}/\mathcal{R} \,, \tag{2.10}$$

i.e., $\mathcal{H}$ is the set of equivalence classes $\{f\}$ of vectors f in $\mathcal{H}_{\rm L}$ whose difference is proportional to k, with inner product

$$\big\langle \{f\}, \{f'\} \big\rangle = \big(f, f' \big) \,. \tag{2.11}$$

Any elements of the two equivalence classes may be used to evaluate the inner product on the right-hand side. The radical $\mathcal{R}$ is itself an equivalence class in $\mathcal{H}_{\rm L}$, which corresponds to the zero vector in $\mathcal{H}$. The inner product in $\mathcal{H}$ is not just nonnegative; it is positive definite. The operations of additon and scalar

[1] A subspace of a linear space with nondefinite metric is called a *radical* if all vectors in the subspace are orthogonal to all vectors in the space. Such subspaces are also called *isotropic* subspaces, because all vectors in a radical necessarily have zero length.

multiplication in $\mathcal{H}$ are well-defined by addition and scalar multiplication of any representations from the equivalence classes. $\mathcal{H}$ is a true Hilbert space.

If we change a one-photon wave function from $\mathcal{H}_{\mathrm{L}}$ by adding an element of the radical $\mathcal{R}$, the corresponding state in the physical space $\mathcal{H}$ is unchanged. This is a partial quantum-mechanical analog of the invariance of the classical **E** and **B** fields under a gauge transformation of the vector potential within the Lorentz gauge class. The physical one-photon states are equivalence classes of wave functions related by gauge transformations of the form

$$f^{\mu}(\mathbf{k}) \;\to\; f^{\mu}(\mathbf{k}) + k^{\mu} g(\mathbf{k}). \tag{2.12}$$

The rather abstract notion of a pseudo Hilbert space $\mathcal{H}_{\mathrm{L}}$ is useful here because it simplifies the discussion of Lorentz invariance, which is a nontrivial consideration for photons. In the next section we shall define the action of the homogeneous Lorentz transformations by an ordinary, vector-field transformation law on $\mathcal{H}_{\mathrm{L}}$. The action on physical states is then defined by "passing to the quotient", $\mathcal{H}$. The way this works is the following. We say that a linear operator U on $\mathcal{H}_{\mathrm{L}}$ is *pseudo unitary* if

(i) U and U^{-1} are defined on all of $\mathcal{H}_{\mathrm{L}}$.

(ii) $(Uf,\, Ug) = (f,\, g)$ for all $f, g \in \mathcal{H}_{\mathrm{L}}$.

It follows that U^{-1} is pseudo unitary, too. Note that U (as well as U^{-1}) leaves the radical invariant:

$$U\mathcal{R} \subset \mathcal{R}\,, \tag{2.13}$$

because if f has zero length, so does Uf. It follows that U induces a *unitary* operator on $\mathcal{H}$ via the formula

$$U\,\{f\} = \{Uf\}\,. \tag{2.14}$$

Proof.

(i) First of all, the mapping $\{f\} \to U\,\{f\}$ is well-defined because $\{Uf\}$ is independent of the representation f. That is,

$$\{f\} = \left\{f'\right\} \quad \Rightarrow \quad f - f' \in \mathcal{R} \quad \Rightarrow \quad U(f - f') \in \mathcal{R}\,, \tag{2.15}$$

hence,

$$\{Uf\} = \left\{Uf + U(f' - f)\right\} = \left\{Uf'\right\}. \tag{2.16}$$

(ii) It is easy to check that the operations $U\,\{f\}$ and $U^{-1}\,\{f\}$ are linear.

(iii) $\langle U\,\{f\},\, U\,\{f\}\rangle = (Uf,\, Uf) = (f,\, f) = \langle\{f\},\, \{f\}\rangle$; thus we have the operator equations on $\mathcal{H}$:

$$U^*U = I\,, \qquad U^* = U^{-1}. \tag{2.17}$$

□

To summarize, a *pseudo-unitary* operator on $\mathcal{H}_\mathrm{L}$ induces a *unitary* operator on $\mathcal{H}$.

One last, technical point, which we shall promptly ignore: if we have a family of pseudo-unitary operators $U(\lambda)$ on $\mathcal{H}_\mathrm{L}$, labeled by some real parameter (or set of parameters) λ, which has matrix elements continuous in λ, then the induced unitary operators $U(\lambda)$ on $\mathcal{H}$ also have continuous matrix elements, because

$$\langle\{f\},\, U(\lambda)\,\{g\}\rangle = (f,\, U(\lambda)g)\,. \tag{2.18}$$

3 Representation of the Poincaré group

The homogeneous Lorentz group, L(4, R), has four connected pieces, which are continuously connected to one of the four transformations in its discrete subgroup, the identity, space inversion, time inversion, or total inversion. The identity component is the proper, orthochronous subgroup, traditionally called $\mathrm{L}_+^\uparrow$.

The inhomogeneous Lorentz group is called the Poincaré group, and has four connected components corresponding to those of the homogeneous group. Its elements consist of pairs (a, Λ), $a \in \mathbb{R}^4$, $\Lambda \in \mathrm{L}(4, \mathrm{R})$, with the multiplication law

$$(a, \Lambda)\,(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda'). \tag{3.1}$$

3.a Continuous transformations

We define the action of the identity component of the Poincaré group on $\mathcal{H}_\mathrm{L}$ by

$$[U(a, \Lambda)f]^\mu\,(\mathbf{k}) = e^{ik\cdot a}\,\Lambda^\mu{}_\nu\, f^\nu(\Lambda^{-1}\mathbf{k}), \tag{3.2}$$

where $\Lambda^{-1}\mathbf{k}$ is defined to be the three-vector part of $\Lambda^{-1}k$. This transformation maps $\mathcal{H}_\mathrm{L}$ onto itself because it preserves the Lorentz condition and the sign of the energy; it is also a representation of the group, and is pseudo unitary because of the invariance of the measure element $d^3p/2\omega$ and the invariant contraction $\bar{f}\cdot f$ in the pseudo inner product.

The representation is continuous in the group parameters, and therefore induces a unitary, continuous representation on the physical space $\mathcal{H}$.

3.b Discrete transformations

The action of the discrete transformations is:

$$\textit{space inversion}: \quad (\mathbb{P}f)^\mu(\mathbf{k}) = \eta_{\mathrm{P}}\, f_\mu(-\mathbf{k}), \quad |\eta_{\mathrm{P}}| = 1\,, \tag{3.3a}$$

$$\textit{time inversion}: \quad (\mathbb{T}f)^\mu(\mathbf{k}) = \eta_{\mathrm{T}}\, \overline{f_\mu(-\mathbf{k})}, \quad |\eta_{\mathrm{T}}| = 1\,, \tag{3.3b}$$

$$\textit{total inversion}: \quad (\mathbb{Y}f)^\mu(\mathbf{k}) = \eta_{\mathrm{Y}}\, \overline{f^\mu(\mathbf{k})}, \quad |\eta_{\mathrm{Y}}| = 1\,, \tag{3.3c}$$

$$\textit{charge conjugation}: \quad (\mathbb{C}f)^\mu(\mathbf{k}) = \eta_{\mathrm{C}}\, f^\mu(\mathbf{k}), \quad |\eta_{\mathrm{C}}| = 1\,. \tag{3.3d}$$

The constant phase factors η_{P}, η_{T}, and η_{C} are arbitrary,[2] while $\eta_{\mathrm{Y}} = \eta_{\mathrm{P}}\,\eta_{\mathrm{T}}$ because $\mathbb{Y} = \mathbb{P}\mathbb{T}$ is required for representation of the discrete symmetry subgroup. Note that $\mathbb{P}$ and $\mathbb{C}$ are pseudo unitary, and that $\mathbb{T}$ and $\mathbb{Y}$ are pseudo antiunitary. Charge conjugation is trivial, because the photon is its own antiparticle.

3.c Infinitesimal generators

We have built in zero mass and positive energy for the photon from the beginning. This is expressed by the action of the four-momentum operator P^μ, which is the infinitesimal generator of the translations:

$$\begin{aligned} U(a, I) &\equiv T(a) = \exp iP\cdot a, \\ (P^\nu f)^\mu(\mathbf{k}) &= k^\nu\, f^\mu(\mathbf{k}), \qquad k^0 = \omega \geq 0\,. \end{aligned} \tag{3.4}$$

The mass-squared operator has only the discrete eigenvalue zero in its spectrum:

$$M^2 \equiv P\cdot P = 0\,. \tag{3.5}$$

The operator P^μ is *pseudo Hermitean*, i.e.,

$$(P^\mu f,\, f) = (f,\, P^\mu f)\,. \tag{3.6}$$

[2] And unobservable for one-particle states.

The infinitesimal generators of the three-dimensional rotations are defined in terms of the parametrization for rotations in $\mathrm{L}_+^\uparrow$:

$$R(\theta, \mathbf{e}) = \exp -i\theta \mathbf{e}\cdot\mathbf{S}\,, \qquad \mathbf{e}\cdot\mathbf{e} = 1\,,$$
$$(\mathbf{S}_i)^\mu{}_\nu = -i\,\epsilon_{0i}{}^\mu{}_\nu\,, \quad i = 1, 2, 3\,, \tag{3.7}$$

where $\epsilon_{0123} = -1$. In other words,

$$\mathbf{S} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{S}^{3\times 3} \end{array}\right), \tag{3.8}$$

where

$$\left(\mathbf{S}_i^{3\times 3}\right)_{jk} = -i\,\epsilon_{ijk}\,. \tag{3.9}$$

The rotation operator on $\mathcal{H}_\mathrm{L}$ is

$$U(0, R) \equiv U(R) = \exp -i\theta\mathbf{e}\cdot\mathbf{J}\,. \tag{3.10}$$

By the standard technique, we find the generators $\mathbf{J}$ by looking at infinitesimal rotations:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}\,,$$
$$\mathbf{L} = -i\mathbf{k} \times \frac{\partial}{\partial \mathbf{k}}\,. \tag{3.11}$$

The orbital and spin parts $\mathbf{L}$ and $\mathbf{S}$ commute, because $\mathbf{L}$ acts only on the momentum argument of a vector f while $\mathbf{S}$ acts only on the four-vector index:

$$(\mathbf{S}f)^\mu(\mathbf{k}) = \mathbf{S}^\mu{}_\nu\, f^\nu(\mathbf{k})\,. \tag{3.12}$$

The boosts are parametrized in $\mathrm{L}_+^\uparrow$ by the rapidity λ along a direction $\mathbf{n}$, $\mathbf{n}\cdot\mathbf{n} = 1$,

$$L(\lambda, \mathbf{n}) = \exp i\lambda\mathbf{n}\cdot\mathbf{K}^\mathrm{S}\,,$$
$$(\mathbf{K}_j^\mathrm{S})^\mu{}_\nu = -i\,(g^{\mu 0}\, g_\nu{}^j - g^{\mu j}\, g_\nu{}^0), \tag{3.13}$$

or in matrix notation:

$$\mathbf{K}^{\mathrm{S}}_1 = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad \mathbf{K}^{\mathrm{S}}_2 = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},$$
$$\mathbf{K}^{\mathrm{S}}_3 = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}. \tag{3.14}$$

The superscript "S" indicates "spin part".

The infinitesimal generators of the boosts on $\mathcal{H}_{\mathrm{L}}$ are defined by

$$U(0, L) \equiv U(L) = \exp i\lambda \mathbf{n}\cdot\mathbf{K}\,. \tag{3.15}$$

By the same standard technique as for the rotations, one finds

$$\mathbf{K} = \mathbf{K}^{\mathrm{L}} + \mathbf{K}^{\mathrm{S}}\,,$$
$$\mathbf{K}^{\mathrm{L}} = i\omega\,\frac{\partial}{\partial\mathbf{k}}\,. \tag{3.16}$$

Again, the orbital and spin parts commute.

Because the finite transformations $U(a, \Lambda)$ preserve $\mathcal{H}_{\mathrm{L}}$, and the Lorentz condition in particular, it follows that the infinitesimal transformations, and hence the infinitesimal generators $\boldsymbol{P}$, $\mathbf{J}$, and $\mathbf{K}$, do, too. That is easy to verify directly, for $\boldsymbol{P}$. To check explicitly for $\mathbf{J}$ and $\mathbf{K}$ involves a small calculation, because the orbital and spin parts do *not* separately preserve $\mathcal{H}_{\mathrm{L}}$; only the *sum* does. A straightforward application of the definitions gives

$$k\cdot(\mathbf{L}f) = -\left[\,\mathbf{L},\, k_\mu\right] f^\mu = [\,\mathbf{L},\, \mathbf{k}\,]\cdot\mathbf{f}$$
$$= -i\,\mathbf{k}\times\mathbf{f}\,, \tag{3.17a}$$

$$k\cdot(\mathbf{S}_j f) = -i\,k_\mu\,\epsilon_{0j}{}^{\mu}{}_{\nu}\,f^\nu$$
$$= i\,(\mathbf{k}\times\mathbf{f})_j\,; \tag{3.17b}$$

so $k\cdot(\mathbf{J}f) = 0$. Similarly

$$k\cdot(\mathbf{K}^{\mathrm{L}} f) = -i \left[\omega\frac{\partial}{\partial\mathbf{k}},\, k\right]\cdot f$$

$$= -i\,\mathbf{k}f^0 + i\,\omega\mathbf{f}\,, \tag{3.18a}$$

$$k\cdot(\mathbf{K}^{\mathrm{S}} f) = i\,\mathbf{k}\,f^0 - i\,\omega\,\mathbf{f}\,; \tag{3.18b}$$

so that $k\cdot(\mathbf{K}f) = 0$.

Thus the infinitesimal generators $\mathbf{J}$ and $\mathbf{K}$ are pseudo Hermitean, just as P is.

We saw earlier that pseudo-unitary operators on $\mathcal{H}_{\mathrm{L}}$ induce unitary operators on $\mathcal{H}$. As a matter of fact the analogous statement is true for pseudo-antiunitary operators. It is also true that pseudo-Hermitean operators on $\mathcal{H}_{\mathrm{L}}$ induce Hermitean operators on $\mathcal{H}$, by the following argument. If a linear operator A is pseudo Hermitean,

$$(Af,\, g) = (f,\, Ag)\,, \tag{3.19}$$

then it preserves the radical,

$$f \in \mathcal{R} \quad \Rightarrow \quad (Af,\, Af) = \left(f,\, A^2 f\right) = 0\,, \tag{3.20}$$

because f is orthogonal to all of $\mathcal{H}_{\mathrm{L}}$, which means that $Af \in \mathcal{R}$. Our earlier argument that pseudo-unitary transformations induce linear transformations on $\mathcal{H}$ depended only on the fact that pseudo-unitary operators preserve the radical. Thus A induces a linear transformation on $\mathcal{H}$, which is Hermitean because

$$\langle A\,\{f\},\, \{g\}\rangle = (Af,\, g) = (f,\, Ag) = \langle\{f\},\, A\,\{g\}\rangle\,. \tag{3.21}$$

It follows that the infinitesimal generators of the Poincaré group induce *observables* on the physical space $\mathcal{H}$.

4 Helicity

Photons are known to have spin one, with only two independent states of polarization. We have built that property in, too; but it takes some work to dig it out.

4.a Observables

The helicity operator on $\mathcal{H}_{\mathrm{L}}$ is

$$\widehat{\mathbf{P}}\cdot\mathbf{S} \equiv \frac{\mathbf{P}}{|\mathbf{P}|}\cdot\mathbf{S} = \widehat{\mathbf{P}}\cdot\mathbf{J} = \mathbf{J}\cdot\widehat{\mathbf{P}}. \tag{4.1}$$

We have used the fact that $\mathbf{P}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{P} = 0$. Although we saw earlier that $\mathbf{S}$ does not preserve the Lorentz condition, the helicity operator does, because $\mathbf{P}$ and $\mathbf{J}$ do. It should also be clear that the helicity is pseudo Hermitean, and hence induces a physical observable.

The important fact that we want to understand is that the physical helicity is a Poincaré invariant. There is a subtlety here; namely, the helicity does *not* commute with the Lorentz transformations on $\mathcal{H}_{\mathrm{L}}$.

It does commute with the translations, because $\mathbf{P}$ and $\mathbf{S}$ do; and it is manifestly a rotation invariant; but it does not commute with the boosts. We can see that for infinitesimal boosts after a little computation to get the commutator with $\mathbf{K}$:

$$\left[\mathbf{K},\, \widehat{\mathbf{P}}\cdot\mathbf{S}\right] = i\left(\mathbf{S} - \widehat{\mathbf{P}}\widehat{\mathbf{P}}\cdot\mathbf{S} + \widehat{\mathbf{P}}\times\mathbf{K}^{\mathrm{S}}\right). \tag{4.2}$$

The r.h.s. is not zero on $\mathcal{H}_{\mathrm{L}}$. A tedious but straightforward calculation of its action yields:

$$\left[\mathbf{K},\, \widehat{\mathbf{P}}\cdot\mathbf{S}\right] f^{\mu} = -i\,\frac{k^{\mu}}{\omega}\,\widehat{\mathbf{k}}\times\mathbf{f}\,. \tag{4.3}$$

Although the commutator is not zero, it does map any vector in $\mathcal{H}_{\mathrm{L}}$ into the radical, as we see on the r.h.s. It follows that the physical commutator induced by the above is indeed zero on the physical space $\mathcal{H}$.

Note that multiple commutators of $\mathbf{n}\cdot\mathbf{K}$ with $\widehat{\mathbf{P}}\cdot\mathbf{S}$, which one would need to compute the formal power series expansion of $U(L)\,\widehat{\mathbf{P}}\cdot\mathbf{S}\,U(L)^{-1}$, also project into the radical, because a multiple commutator

$$(\mathrm{ad}\,\mathbf{n}\cdot\mathbf{K})^{\ell}\,\widehat{\mathbf{P}}\cdot\mathbf{S} \tag{4.4}$$

can be written as a sum of monomials of the form

$$(\mathbf{n}\cdot\mathbf{K})^{m}\left[\mathbf{n}\cdot\mathbf{K},\, \widehat{\mathbf{P}}\cdot\mathbf{S}\right](\mathbf{n}\cdot\mathbf{K})^{n}, \quad m+n = \ell-1\,. \tag{4.5}$$

When the commutator acts, we are left in the radical; and we stay there when $(\mathbf{n}\cdot\mathbf{K})^{m}$ acts, because $\mathbf{n}\cdot\mathbf{K}$ is pseudo Hermitean and preserves the radical. Thus the conjugated helicity operator has an action of the form

$$\left[U(L)\,\widehat{\mathbf{P}}\cdot\mathbf{S}\,U(L)^{-1}f\right]^{\mu} = \left(\widehat{\mathbf{P}}\cdot\mathbf{S}\,f\right)^{\mu} + k^{\mu}\,g\,. \tag{4.6}$$

where g can be computed, but does not interest us now.

For the boosts on the physical space, this just confirms what we already know from the fact that helicity commutes with the physical $\mathbf{K}$: the physical helicity is a Lorentz invariant, in fact a Poincaré invariant.

The discrete transformations pose no problem. A straightforward computation gives the expected results:

$$\mathbb{P}\,\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,\mathbb{P}^{-1} = -\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,, \tag{4.7a}$$

$$\mathbb{T}\,\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,\mathbb{T}^{-1} = \widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,, \tag{4.7b}$$

$$\mathbb{Y}\,\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,\mathbb{Y}^{-1} = -\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,, \tag{4.7c}$$

$$\mathbb{C}\,\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,\mathbb{C}^{-1} = \widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,. \tag{4.7d}$$

For future reference, we note the action of the helicity operator:

$$\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,f = \big(0, i\,\widehat{\mathbf{k}}\times\mathbf{f}\big). \tag{4.8}$$

4.b States

Now that we know helicity to be a Poincaré invariant,[3] we expect the physical space $\mathcal{H}$ to split into a direct sum of representation spaces for the Poincaré group, labeled by the eigenvalues of helicity. Our aim is to study the splitting of $\mathcal{H}_{\mathrm{L}}$ into helicity eigenspaces, which induces the invariant splitting of $\mathcal{H}$.

To find the helicity eigenstates, we use the identity

$$\big(\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\big)^3 = \widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\,, \tag{4.9}$$

which is easy to check, and which is a property of the spin-one representation of $\mathbf{S}$. The quantity

$$E_{\mathrm{tr}} \equiv \big(\widehat{\mathbf{P}}\!\cdot\!\mathbf{S}\big)^2 \tag{4.10}$$

is therefore a projection operator. It plays a special role in the theory of photons; it is just the projection operator for transverse polarization, or the projection operator for the radiation gauge:

$$E_{\mathrm{tr}}\,f = \big(0, \mathbf{f} - \widehat{\mathbf{k}}\,\widehat{\mathbf{k}}\!\cdot\!\mathbf{f}\big), \qquad \big(E_{\mathrm{tr}}\big)^2 = E_{\mathrm{tr}}\,. \tag{4.11}$$

[3] When we say "Poincaré", we always mean the restricted Poincaré group, without inversions, unless we say otherwise.

The identity in Eq. (4.9) can also be written

$$E_{\text{tr}}\,\hat{\mathbf{P}}\cdot\mathbf{S} = \hat{\mathbf{P}}\cdot\mathbf{S}\,E_{\text{tr}} = \hat{\mathbf{P}}\cdot\mathbf{S}\,. \tag{4.12}$$

The projection operators for helicity ± 1 are

$$E_{\pm} = \tfrac{1}{2}\big(E_{\text{tr}} \pm \hat{\mathbf{P}}\cdot\mathbf{S}\big), \tag{4.13}$$

as one readily verifies that

$$\begin{gathered}\hat{\mathbf{P}}\cdot\mathbf{S}\,E_{\pm} = \pm E_{\pm}\,, \qquad E_{\pm}\,E_{\pm} = E_{\pm}\,, \qquad E_{\pm}E_{\mp} = 0\,,\\ E_{\pm}\,f \equiv f_{\pm} = \tfrac{1}{2}\big(0, \mathbf{f} - \hat{\mathbf{k}}\hat{\mathbf{k}}\cdot\mathbf{f} \pm i\,\hat{\mathbf{k}}\times\mathbf{f}\big).\end{gathered} \tag{4.14}$$

The helicity eigenstates take on a familiar form if we recognize that, since $\mathbf{k}\cdot\mathbf{f}_{\pm} = 0$, $f_{\pm}$ must be linear combinations of the two, purely spatial basis vectors e_{λ} introduced in Section 2. Suppose we chose $\mathbf{e}_1$, $\mathbf{e}_2$, and $\hat{\mathbf{k}}$ to form a right-handed, orthonormal set. Then it is easy to check that $f_{\pm}$ have the form

$$f^{\mu}_{\pm} = e^{\mu}_{\pm}\,g_{\pm}\,, \qquad e_{\pm} \equiv \frac{e_1 \pm i\,e_2}{\sqrt{2}}\,. \tag{4.15}$$

If we were discussing the classical, free electromagnetic field, these would be the polarization vectors for left and right circular polarization.

The above form is independent of the choice of $\mathbf{e}_1$ and $\mathbf{e}_2$, because a rotation about $\hat{\mathbf{k}}$ just multiplies $\mathbf{e}_{\pm}$ by phase factors:

$$\exp(-i\,\theta\,\hat{\mathbf{P}}\cdot\mathbf{S})\,e_{\pm} = e^{\mp i\theta}\,e_{\pm}\,. \tag{4.16}$$

So far, we have pulled out two pieces of $\mathcal{H}_{\text{L}}$, the helicity ± 1 subspaces

$$\mathcal{H}_{\text{L},\pm 1} = E_{\pm}\,\mathcal{H}_{\text{L}}\,. \tag{4.17}$$

But not every vector in $\mathcal{H}_{\text{L}}$ is a linear combination of these. If we go back to the action of the helicity operator in Eq. (4.8), it is easy to see that the radical consists of *zero*-helicity eigenvectors, because then $\mathbf{f}$ is proportional to $\mathbf{k}$. We also note that all zero-helicity vectors belong to the radical, for

$$\hat{\mathbf{P}}\cdot\mathbf{S}\,f = 0 \quad\Rightarrow\quad \mathbf{k}\times\mathbf{f} = 0 \quad\Rightarrow\quad \mathbf{f} = \mathbf{k}\,g\,, \tag{4.18}$$

and the Lorentz condition then implies that $f^0 = \omega g$.

That takes care of the rest of $\mathcal{H}_\mathrm{L}$, because we saw in Section 2 that all vectors in $\mathcal{H}_\mathrm{L}$ could be expanded in terms of k and the two e_λ. The zero-helicity projection operator is therefore

$$\begin{aligned} E_0 &= I - E_+ - E_- , \\ E_0\, f &= \frac{k\,\tilde{k}\cdot f}{2\omega^2} . \end{aligned} \tag{4.19}$$

We have thus arrived at a decomposition of $\mathcal{H}_\mathrm{L}$ into three pieces:

$$\mathcal{H}_\mathrm{L} = \mathcal{H}_{\mathrm{L},0} \oplus \mathcal{H}_{\mathrm{L},+1} \oplus \mathcal{H}_{\mathrm{L},-1} . \tag{4.20}$$

The pieces are guaranteed to be orthogonal, because the standard argument that gives the orthogonality of eigenstates of a Hermitean operator belonging to different eigenvalues works just as well for pseudo-Hermitean operators.

The zero-helicity space, since it is the radical, goes away in the physical space $\mathcal{H}$. The decomposition of $\mathcal{H}_\mathrm{L}$ induces an orthogonal splitting of $\mathcal{H}$ into two pieces:

$$\mathcal{H} = \mathcal{H}_{+1} \oplus \mathcal{H}_{-1} . \tag{4.21}$$

The action of the Poincaré group on each helicity eigenspace can be shown to be irreducible, so we have found that the physical photon representation is a direct sum of two irreducible representations, one for each physical value of the helicity. This property persists for zero-mass particles with any discrete spin S; the irreducible representations are labeled by helicity, which can have only two values, $\pm S$.

We claimed in Eq. (4.7a) that helicity anticommutes with space inversion. It follows that $\mathbb{P}$ interchanges the two ± 1 helicity spaces. That should be clear by inspection of the helicity eigenstates in Eq. (4.14); the relative sign of the vector and axial vector parts of $f_\pm$ is changed under space inversion. Time inversion commutes with helicity, and so does not mix the two spaces. The relative sign of the vector and axial vector parts of $f_\pm$ does not change sign under $\mathbb{T}$ because, although $\mathbf{k} \to -\mathbf{k}$, the factor i in the axial part anticommutes with an antiunitary operator.

5 Realization of the physical space

Although the abstract formulation for the physical space $\mathcal{H}$ is quite adequate for practical problems involving photons, it is pedagogically valuable to have a concrete realization. We do this in two ways. In the first, we choose to work in one of the more popular and useful gauges, the radiation gauge. The second makes helicity explicit.

5.a Radiation gauge realization

Picking a gauge amounts to picking a representative f from each equivalence class $\{f\}$. We get the radiation gauge if we choose that representative which obeys the transversality condition:

$$f^0 = 0\,, \qquad \mathbf{k}\cdot\mathbf{f} = 0\,, \tag{5.1}$$

or in four-vector notation,

$$k\cdot f = \widetilde{k}\cdot f = 0\,. \tag{5.2}$$

Note that there is indeed precisely one transverse member of each class, because every $f \in \mathcal{H}_{\mathrm{L}}$ has the expansion

$$f^\mu = k^\mu\, g_\parallel + \textstyle\sum_\lambda e^\mu_\lambda\, g_\lambda\,; \tag{5.3}$$

and the transverse member of the class $\{f\}$ is just

$$E_{\mathrm{tr}}\, f = \textstyle\sum_\lambda e_\lambda\, g_\lambda\,. \tag{5.4}$$

The vectors e_λ describe the transverse polarization states of the photon, and we could just as well use helicity polarization vectors $e_\pm$.

We now have a one-to-one, linear, norm-preserving correspondence between the elements $\{f\}$ of the physical space $\mathcal{H}$ and the members f_{tr} of the subspace of $\mathcal{H}_{\mathrm{L}}$:

$$\mathcal{H}_{\mathrm{tr}} = E_{\mathrm{tr}}\,\mathcal{H}_{\mathrm{L}} = \mathcal{H}_{\mathrm{L},+1} \oplus \mathcal{H}_{\mathrm{L},-1} \tag{5.5}$$

The correspondence is $\{f\} \leftrightarrow E_{\mathrm{tr}} f$. The space $\mathcal{H}_{\mathrm{tr}}$ is the realization of the physical space $\mathcal{H}$ in the radiation gauge.

To compute the action of the Poincaré group in the radiation gauge, we define

$$U_{\text{tr}}(a,\Lambda) = E_{\text{tr}}\, U(a,\Lambda)\, E_{\text{tr}}\,. \tag{5.6}$$

These operators act entirely within $\mathcal{H}_{\text{tr}}$, and they are carried over by the correspondence between $\mathcal{H}_{\text{tr}}$ and $\mathcal{H}$ into the physical representation of the Poincaré group. In fact, the matrix elements of U_{tr} in $\mathcal{H}_{\text{tr}}$ are equal to the matrix elements of U in $\mathcal{H}$:

$$\big(f,\, U_{\text{tr}}(a,\Lambda)\, g\big) = \big(E_{\text{tr}}\, f,\, U(a,\Lambda)\, E_{\text{tr}}\, g\big) = \langle\{f\},\, U(a,\Lambda)\,\{g\}\rangle\,. \tag{5.7}$$

In particular, it follows from this formula and the fact that the physical $U(a,\Lambda)$ are a representation of the group that the operators $U_{\text{tr}}(a,\Lambda)$ obey the multiplication law of the Poincaré group, a fact which seems less obvious when we recall that the projection operator E_{tr} does not commute with $U(a,\Lambda)$ in $\mathcal{H}_{\text{L}}$, because boosts can mix in some zero helicity.

The group multiplication law can be derived by the following direct argument.[4] First, note that

$$E_{\text{tr}}\, U(a,\Lambda)\, E_{\text{tr}} = E_{\text{tr}}\, U(a,\Lambda)\,, \tag{5.8}$$

because for any f in $\mathcal{H}_{\text{L}}$ we have $f = kg_0 + f_{\text{tr}}$; and since $U(a,\Lambda)$ preserves the radical,

$$E_{\text{tr}}\, U(a,\Lambda)\, kg_0 = 0\,, \tag{5.9}$$

Therefore

$$\begin{aligned} E_{\text{tr}}\, U(a,\Lambda)\, E_{\text{tr}}\, U(a',\Lambda')\, E_{\text{tr}} &= E_{\text{tr}}\, U(a,\Lambda)\, U(a',\Lambda') \\ &= E_{\text{tr}}\, U(a+\Lambda a',\Lambda\Lambda') \\ &= E_{\text{tr}}\, U(a+\Lambda a',\Lambda\Lambda')\, E_{\text{tr}}\,. \end{aligned} \tag{5.10}$$

We can describe the effects of the three factors in U_{tr} as follows. First, E_{tr} acts on a state f to select the radiation gauge. Then $U(a,\Lambda)$ corresponds to a conventional transformation law for a four-vector field. At this point, we are still in the Lorentz gauge class, i.e., still in $\mathcal{H}_{\text{L}}$, but no longer generally in the radiation

[4] This remark was contributed by Prof. Andreas Blass from the mathematics department, who audited the course.

gauge. The final factor E_{tr} is a gauge transformation that restores the radiation gauge. Equation (5.8) shows that the result is the same, even when the radiation gauge is not selected first.

The action of U_{tr} can be written explicitly with the help of the gauge transformation formula

$$E_{\text{tr}}\, f^{\mu} = (I - E_0)\, f^{\mu} = f^{\mu} - \frac{k^{\mu}\,\widetilde{k}\cdot f}{2\omega^2}\,. \tag{5.11}$$

Then, whether f is in $\mathcal{H}_{\text{tr}}$ or $\mathcal{H}_{\text{L}}$ we get

$$U_{\text{tr}}(a,\Lambda)\, f(\mathbf{k}) = e^{ik\cdot a}\left[\Lambda\, f(\Lambda^{-1}\mathbf{k}) - \frac{k\,\widetilde{k}\cdot\Lambda f(\Lambda^{-1}\mathbf{k})}{2\omega^2}\right]. \tag{5.12}$$

This transformation law is of course not as simple as that of a four-vector field, and it looks even more awkward if we write it as a three-vector transformation law relating the nonvanishing components of f and $U_{\text{tr}} f$.

As we defined them earlier, space and time inversion already preserve the radiation gauge, and so do not require a special discussion.

The complication in the Poincaré transformations can be thrown into a phase factor if we choose helicity polarization vectors $e_{\pm}(\mathbf{k})$. The invariance of the helicity eigenspaces says that

$$U_{\text{tr}}(a,\Lambda)\, f_{\pm}(\mathbf{k}) = e_{\pm}(\mathbf{k})\, g_{\pm}^{(a,\Lambda)}(\mathbf{k}), \tag{5.13}$$

while the definition of the transformation law says

$$U_{\text{tr}}(a,\Lambda)\, f_{\pm}(\mathbf{k}) = e^{ik\cdot a}\, E_{\text{tr}}\, \Lambda e_{\pm}(\Lambda^{-1}\mathbf{k})\, g_{\pm}(\Lambda^{-1}\mathbf{k}). \tag{5.14}$$

Now the norms are[5]

$$\left(f_{\pm},\, f_{\pm}\right) = \int \frac{\mathrm{d}^3 k}{2\omega}\, \left|g_{\pm}(\mathbf{k})\right|^2 \tag{5.15}$$

$$= \left(U_{\text{tr}}(a,\Lambda)\, f_{\pm},\, U_{\text{tr}}(a,\Lambda)\, f_{\pm}\right) \tag{5.16}$$

$$= \int \frac{\mathrm{d}^3 k}{2\omega}\, \left|g_{\pm}^{(a,\Lambda)}(\mathbf{k})\right|^2 \tag{5.17}$$

$$= \int \frac{\mathrm{d}^3 k}{2\omega}\, \left|g_{\pm}(\Lambda^{-1}\mathbf{k})\right|^2, \tag{5.18}$$

[5] Note that $\overline{e_{\pm}(\mathbf{k})}\cdot e_{\pm}(\mathbf{k}) = -1$.

where the last line follows because

$$E_{\rm tr}\,\Lambda e_\pm(\Lambda^{-1}\mathbf{k}) = \Lambda e_\pm(\Lambda^{-1}\mathbf{k}) - \frac{k\,\widetilde{k}\cdot\Lambda e_\pm(\Lambda^{-1}\mathbf{k})}{2\omega^2}\,; \tag{5.19}$$

and so,

$$\begin{aligned}&\overline{E_{\rm tr}\,\Lambda e_\pm(\Lambda^{-1}\mathbf{k})}\cdot E_{\rm tr}\,\Lambda e_\pm(\Lambda^{-1}\mathbf{k})\\ &\quad = \overline{\Lambda e_\pm(\Lambda^{-1}\mathbf{k})}\cdot \Lambda e_\pm(\Lambda^{-1}\mathbf{k}) = -1\,.\end{aligned} \tag{5.20}$$

Since the three integrals are equal for all $g_\pm$, and since $E_{\rm tr}\,\Lambda e_\pm(\Lambda^{-1}\mathbf{k})$ must be proportional to $e_\pm(\mathbf{k})$, we conclude that

$$E_{\rm tr}\,\Lambda e_\pm(\Lambda^{-1}\mathbf{k}) = e^{i\phi_\pm(\mathbf{k},\Lambda)}\,e_\pm(\mathbf{k}), \tag{5.21}$$

and

$$g_\pm^{(a,\Lambda)}(\mathbf{k}) = e^{ik\cdot a}\,e^{i\phi_\pm(\mathbf{k},\Lambda)}\,g_\pm(\Lambda^{-1}\mathbf{k}), \tag{5.22}$$

where the phase $\phi_\pm(\mathbf{k},\Lambda)$ can be computed, once we establish a phase convention for the definition of $e_\pm(\mathbf{k})$. We shall not bother with that here; we just mention that the general representation theory of the Poincaré group provides a natural way to do it.

5.b Helicity realization

We have already recognized that the physical space splits into two, Poincaré invariant, helicity eigenspaces. By choosing a helicity basis for the radiation gauge wave functions, we get a second realization of the physical space as

$$\mathcal{H}_{\rm hel} = \mathrm{L}^2\left(\frac{\mathrm{d}^3k}{2\omega}\right)\otimes\mathbb{C}^2 = \mathrm{L}^2\left(\frac{\mathrm{d}^3k}{2\omega}\right)\oplus\mathrm{L}^2\left(\frac{\mathrm{d}^3k}{2\omega}\right), \tag{5.23a}$$

i.e.,

$$\mathcal{H}_{\rm hel} = \left\{ g_\sigma(\mathbf{k}):\ \sigma=\pm,\quad \langle g,\,g\rangle = \textstyle\sum_\sigma \displaystyle\int \frac{\mathrm{d}^3k}{2\omega}\,\left|g_\sigma\right|^2 < \infty \right\}. \tag{5.23b}$$

The one-to-one correspondence between $\mathcal{H}_{\rm tr}$ and $\mathcal{H}_{\rm hel}$ is defined by

$$f = e_+\,g_+ + e_-\,g_-\,,\qquad g_\pm = \overline{e_\pm}\cdot f\,, \tag{5.24a}$$

$$(f,\,f) = \textstyle\sum_\sigma \displaystyle\int \frac{\mathrm{d}^3k}{2\omega}\,\left|g_\sigma\right|^2. \tag{5.24b}$$

The representation of the Poincaré group on $\mathcal{H}_{\text{hel}}$ is given by

$$\big[U_{\text{hel}}(a,\Lambda)\, g\big]_{\pm}(\mathbf{k}) = e^{ik\cdot a}\, e^{i\phi_{\pm}(\mathbf{k},\Lambda)}\, g_{\pm}(\Lambda^{-1}\mathbf{k})\,. \tag{5.25}$$