Precession of the Perihelion of Mercury in Special and General Relativity*

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Abstract

This is a LaTeX version of slides for two lectures I gave for a bag lunch journal club. The first summarizes the lore on the precession of Mercury in special relativity, including my own calculations, in the nature of extended homework. The second summarizes Feynman's calculation, which illustrated his groundbreaking approach to general relativity as a flat-space gauge theory in his *Lectures on Gravitation*.

*January 21, 2009: Aside from the addition of the abstract in September, 2007, this LaTeX version has only cosmetic changes from the original, handwritten slides.

Lectures

Lecture 1

Special Relativity

1 Introduction

context

- Any perturbation which doesn't turn the Kepler Hamiltonian into a harmonic oscillator will cause Kepler ellipses to precess.
- There's nothing wrong with the theory we're about to discuss, it's just not right!
- Feynman's *Lectures on Gravitation*[1]

criteria

- covariance under special relativity
- Newtonian limit
- coupling to energy (Eötvös, binding energy, temperature)

early attempts

- Einstein

[1] Richard P. Feynman, Fernando B. Morinigo, William G. Wagner, *Feynman Lectures on Gravitation*, (Westview Press, Boulder, Colorado, 2003), Brian Hatfield, ed. This is an augmented version of the typewritten notes available at the time of this lecture.

- Sommerfeld
- others?

2 Gravitational potential

2.a Notation

$$x \equiv (ct, \boldsymbol{r}) \equiv x^\mu$$
$$x(t) = (ct, \boldsymbol{r}(t)) = y(\tau)$$
$$\gamma \equiv \left(1 - \beta^2\right)^{-\frac{1}{2}} \qquad \beta \equiv \frac{v}{c}$$
$$\mathrm{d}x^0 = c\,\mathrm{d}t \qquad \mathrm{d}y^0 = \gamma c\,\mathrm{d}\tau$$
$$\boldsymbol{v} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \qquad u \equiv \frac{\mathrm{d}y}{\mathrm{d}\tau} = \gamma(c, \boldsymbol{v})$$
$$a \equiv \frac{\mathrm{d}u}{\mathrm{d}t}$$

2.b Gravitational sources

scalar

$$\rho = \frac{m}{\gamma}\,\delta[\boldsymbol{r} - \boldsymbol{r}(t)] = \int \delta[x - y(\tau)]\, mc\,\mathrm{d}\tau$$

vector

$$j^\mu = \frac{m}{\gamma}\,u^\mu\,\delta[\boldsymbol{r} - \boldsymbol{r}(t)] = \int \delta[x - y(\tau)]\, m\,u^\mu c\,\mathrm{d}\tau$$
$$\partial_\mu j^\mu = 0$$

tensor

$$K^{\mu\nu} = \frac{m}{\gamma}\,u^\mu u^\nu\,\delta[\boldsymbol{r} - \boldsymbol{r}(t)] = \int \delta[x - y(\tau)]\, m\,u^\mu u^\nu c\,\mathrm{d}\tau$$
$$\partial_\mu K^{\mu\nu} = \frac{m}{\gamma}\,a^\nu\,\delta[\boldsymbol{r} - \boldsymbol{r}(t)]$$
$$K^\sigma{}_\sigma = \rho c^2$$

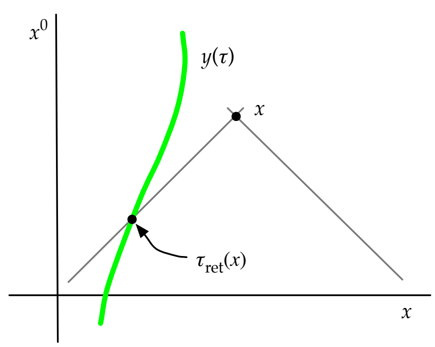
2.c Potentials

$$\Box\phi^{\mu\nu} = -\frac{4\pi G}{c^2}\,K^{\mu\nu}$$

$$\Box\phi^{\mu} = -\frac{4\pi G}{c}\,j^{\mu}$$

$$\Box\phi = -4\pi G\rho = -\frac{4\pi G}{c^2}K^{\sigma}{}_{\sigma}$$

$$\Box\mathcal{G}_{\rm ret} = -\delta(x) \qquad \mathcal{G}_{\rm ret} = \frac{1}{2\pi}\,\theta(x^0)\,\delta(x\cdot x)$$



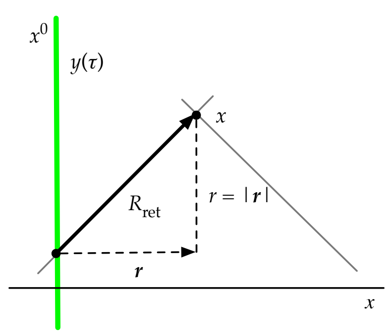
$$R \equiv x - y(\tau) \qquad \text{ret} \Rightarrow \tau = \tau_{\rm ret}(x)$$

$$\phi^{\mu\nu} = -\frac{Gm}{c}\left[\frac{u^\mu u^\nu}{R\cdot u}\right]_{\rm ret}$$

$$\phi^{\mu} = -Gm\left[\frac{u^\mu}{R\cdot u}\right]_{\rm ret}$$

$$\phi = -Gmc\left[\frac{1}{R\cdot u}\right]_{\rm ret}$$

2.d Sun



$$y(\tau) = (ct, \mathbf{0})$$

$$u(\tau) = (c, \mathbf{0})$$

$$R_{\rm ret} = (r, \boldsymbol{r})$$

$$(R\cdot u)_{\rm ret} = cr \qquad m = M$$

$$\phi^{\mu\nu} = -\frac{GM}{r}\,g^{\mu 0}g^{\nu 0}$$

$$\phi^{\mu} = -\frac{GM}{r}\,g^{\mu 0}$$

$$\phi = -\frac{GM}{r}$$

3 Lagrangian

3.a Interaction

$$U_{\mathrm{S}} = \int \phi\,\rho\,\mathrm{d}^3x = -\frac{1}{\gamma}\,\frac{GMm}{r}$$

$$U_{\mathrm{V}} = \int \phi^{\mu}\,\frac{j_{\mu}}{c}\,\mathrm{d}^3x = -\frac{GMm}{r}$$

$$U_{\mathrm{T}} = \int \phi^{\mu\nu}\,\frac{K_{\mu\nu}}{c^2}\,\mathrm{d}^3x = -\gamma\,\frac{GMm}{r}$$

Note: $$U_{\mathrm{S}} = \int g^{\mu\nu}\phi\,\frac{K_{\mu\nu}}{c^2}\,\mathrm{d}^3x$$

3.b Lagrangian

$$L = -\frac{mc^2}{\gamma} - U \qquad \text{(Note: } \gamma = \gamma \text{ of Mercury)}$$

4 Calculation

4.a Expansion

We refer everthing to vector, because U_{V} is velocity independent and gives a simple force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,m\gamma\boldsymbol{v} = -\frac{GMm}{r^3}\,\boldsymbol{r}$$

Let

$$U = -\frac{GMm}{r}\,\alpha(\gamma) \equiv -\frac{\lambda\alpha}{r} \qquad \lambda \equiv GMm$$

$$\begin{aligned} \alpha(\gamma) &= \alpha(1) + (\gamma-1)\alpha'(1) + \cdots \\ &= 1 + (\gamma-1)\chi + \cdots \qquad \alpha(1) = 1 \qquad \alpha'(1) \equiv \chi \\ &= 1 + \frac{\beta^2\chi}{2} + \cdots \end{aligned}$$

$$L = -mc^2 + \frac{mv^2}{2} + \frac{mv^4}{8c^2} + \frac{\lambda}{r} + \frac{mv^2}{2}\xi + \cdots$$
$$= -mc^2 + L_{\text{Kep}} + \frac{mv^4}{8c^2} + \frac{mv^2}{2}\xi + \cdots$$

$$\boxed{\begin{aligned} \xi &\equiv \frac{\lambda\chi}{mrc^2} = \frac{r_0\chi}{r} \\ r_0 &\equiv \frac{GM}{c^2} = \text{grav. radius} \\ &= \tfrac{1}{2}r_s \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{r_0}{r} &= O(\beta^2) \\ \left\langle \frac{mv^2}{2} \right\rangle &= \frac{1}{2}\left\langle \frac{\lambda}{r} \right\rangle \text{ on Kepler orbit} \end{aligned}}$$

4.b Perturbing force

$\Rightarrow$ (to order v^2/c^2)

$$m\boldsymbol{a} = -\frac{\lambda \boldsymbol{r}}{r^3} + \boldsymbol{f}$$

$$\boxed{\boldsymbol{f} = -\frac{\lambda \boldsymbol{r}}{r^3}\left(\frac{\chi - 1}{2}\beta^2 - \frac{r_0}{r}\chi\right) + \frac{\lambda \boldsymbol{r}\cdot\boldsymbol{v}\boldsymbol{v}}{r^3c^2}(1+\chi)}$$

4.c Runge-Lenz vector

$$\boldsymbol{R} = \frac{\boldsymbol{p}\times\boldsymbol{L}}{m} - \frac{\lambda\boldsymbol{r}}{r} = m\boldsymbol{v}\times(\boldsymbol{r}\times\boldsymbol{v}) - \frac{\lambda\boldsymbol{r}}{r}$$
$$\dot{\boldsymbol{R}} = \boldsymbol{f}\times(\boldsymbol{r}\times\boldsymbol{v}) + \boldsymbol{v}\times(\boldsymbol{r}\times\boldsymbol{f})$$
$$\boldsymbol{\Omega} = \frac{\boldsymbol{R}\times\dot{\boldsymbol{R}}}{R^2} \propto \boldsymbol{L} \qquad \boxed{\text{compute } \boldsymbol{\Omega} \text{ over a Kepler orbit}}$$

4.d Other ways to compute

(i) Hamiltonian with P.B.'s. Be careful to reexpress $\boldsymbol{R}$ in terms of relativistic $\boldsymbol{p}$.

(ii) Action-angle variables and Hamilton-Jacobi theory.

(iii) Perturbation of the orbit equation (cleanest).

5 Results

$$\langle \Omega \rangle = (1+2\chi)\,\langle \Omega_{\rm V} \rangle = (1+2\chi)\,\frac{\Omega_{\rm GR}}{6}$$

$$\langle \Omega_{\rm V} \rangle = \frac{\omega}{2}\,\frac{r_0}{a}\,\frac{1}{1-e^2} \qquad \omega = \frac{2\pi}{T} \qquad \begin{array}{l} a = \text{semimajor axis} \\ e = \text{eccentricity} \end{array}$$

Mercury: $\Omega_{\rm GR} = 42.98''/\text{cent}$

$\Omega_{\rm exp} = (41.4 \pm 0.9)''/\text{cent}$ [out of $(5,599.74 \pm 0.41)''/\text{cent}$]

5.a Basic types

vector: $\alpha(\gamma) = 1 \qquad \alpha'(\gamma) = 0 \qquad \chi = 0$

$$\boxed{\Omega_{\rm V} = 7.16''/\text{cent} = \tfrac{1}{6}\,\Omega_{\rm GR}}$$

scalar: $\alpha(\gamma) = \dfrac{1}{\gamma} \qquad \alpha'(\gamma) = -\dfrac{1}{\gamma^2} \qquad \chi = -1$

$$\boxed{\Omega_{\rm S} = -7.16''/\text{cent} = -\Omega_{\rm V}}$$

tensor: $\alpha(\gamma) = \gamma \qquad \alpha'(\gamma) = 1 \qquad \chi = +1$

$$\boxed{\Omega_{\rm T} = 3\,\Omega_{\rm V} = \tfrac{1}{2}\,\Omega_{\rm GR}}$$

5.b Combined types

The combination

$$h^{\mu\nu} = a\,\phi^{\mu\nu} + (1-a)g^{\mu\nu}\phi$$

has the right Newtonian limit, and produces the precession:

$$\boxed{\Omega_a \equiv a\,\Omega_{\rm T} + (1-a)\Omega_{\rm S} = \frac{3a + a - 1}{6}\,\Omega_{\rm GR} = \frac{4a-1}{6}\,\Omega_{\rm GR}}$$

symmetric, traceless tensor: $a - 1 = \dfrac{a}{4} \qquad a = \dfrac{4}{3}$

$$\boxed{\Omega_{\text{ST}} = \frac{\frac{16}{3}-1}{6}\,\Omega_{\text{GR}} = \tfrac{13}{18}\,\Omega_{\text{GR}}}$$

spin 2 tensor: $a - 1 = \dfrac{a}{2} \qquad a = 2$

$$\boxed{\Omega_2 = \tfrac{8-1}{6}\,\Omega_{\text{GR}} = \tfrac{7}{6}\,\Omega_{\text{GR}}} \qquad\qquad \boxed{\Omega_{\text{Feynman}} \approx \tfrac{4}{3}\,\Omega_{\text{GR}}}$$

exact tensor: $4a - 1 - 6 \qquad a = \dfrac{7}{4} \qquad 1 - a = -\dfrac{3}{4}$

$$\boxed{\Omega_{\text{GR}} = \tfrac{7}{4}\,\Omega_{\text{T}} - \tfrac{3}{4}\,\Omega_{\text{S}} = \tfrac{7}{4}\,(\Omega_{\text{T}} - \tfrac{3}{7}\,\Omega_{\text{S}})}$$

6 Feynman's approach

Up to now $g_{\mu\nu}$ has been used for the flat space metric. From now on we use $\eta_{\mu\nu}$ for the flat space metric and $g_{\mu\nu}$ for the metric with gravity. The raising and lowering of indices is defined with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.

6.a Gauge condition

The field $h^{\mu\nu}$ is required to obey the gauge condition:

$$\partial_\mu \overline{h}^{\mu\nu} = 0 \qquad \overline{h}^{\mu\nu} \equiv h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\,h \qquad h \equiv h^\sigma{}_\sigma$$

This is chosen to make $\partial_\mu T^{\mu\nu} = 0$ as a consequence of the field equation, which makes spin 2:

$$\Box h^{\mu\nu} = -\frac{4\pi G}{c^2}\cdot 2\,\overline{T}^{\mu\nu}$$

Note that $\overline{\overline{T}}^{\mu\nu} = T^{\mu\nu}$. Thus:

$$\Box \overline{h}^{\mu\nu} = -\frac{8\pi G}{c^2}\,T^{\mu\nu} \quad \text{and} \quad \partial_\mu \overline{h}^{\mu\nu} = 0 \qquad \Rightarrow \qquad \partial_\mu T^{\mu\nu} = 0$$

For sure the solution for $h^{\mu\nu}$ with $T^{\mu\nu} = K^{\mu\nu}$ does *not* obey the gauge condition.

6.b Invariant Lagrangian

$$S = -\tfrac{1}{2}\int m\, x' \cdot x'\, \mathrm{d}\alpha - \int m\, h_{\mu\nu}\, x'^{\mu} x'^{\nu} \mathrm{d}\alpha$$

$$x' \equiv \frac{\mathrm{d}x}{\mathrm{d}\alpha} \qquad \alpha = \text{“time parameter”}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{c^2}\, h_{\mu\nu}$$

$$\mathbb{L} = -\frac{m}{2}\, x' \cdot x' - \frac{m}{c^2}\, h_{\mu\nu}\, x'^{\mu} x'^{\nu} = -\frac{m}{2}\, g_{\mu\nu}\, x'^{\mu} x'^{\nu}$$

The equations of motion become

$$g_{\mu\nu}\, x''^{\nu} = -[\,\rho\sigma,\, \mu\,]\, x'^{\rho} x'^{\sigma} \qquad [\,\rho\sigma,\, \mu\,] \equiv \tfrac{1}{2}\left[\partial_{\sigma} g_{\rho\mu} + \partial_{\rho} g_{\sigma\mu} - \partial_{\mu} g_{\rho\sigma}\right]$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(g_{\mu\nu}\, x'^{\mu} x'^{\nu}\right) = 0 \qquad \Rightarrow \qquad \text{can choose}$$

$$\boxed{g_{\mu\nu}\, x'^{\mu} x'^{\nu} \equiv \left(\frac{\mathrm{d}s}{\mathrm{d}\alpha}\right)^2 = c^2}$$

Then $s/c = \alpha =$ “proper time”, but including the gravitational field.

Lecture 2

General Relativity

1 Summary of Lecture 1

$$\left\{\begin{matrix}\rho\\ j^\mu\\ K^{\mu\nu}\end{matrix}\right\} = \int \delta[x-y(\tau)]\,\left\{\begin{matrix}M\\ M\,u^\mu\\ M\,u^\mu u^\nu\end{matrix}\right\} c\mathrm{d}\tau$$

$$\Box\left\{\begin{matrix}\phi\\ \phi^\mu\\ \phi^{\mu\nu}\end{matrix}\right\} = -4\pi G\left\{\begin{matrix}\rho\\ j^\mu/c\\ K^{\mu\nu}/c^2\end{matrix}\right\}$$

$$\boxed{\begin{aligned} u &= \frac{\mathrm{d}x}{\mathrm{d}\tau}\\ u\cdot u &= c^2\\ \tau &= \text{proper time}\end{aligned}}$$

$$\begin{aligned}
U_\mathrm{S} &= \int \phi\,\rho\,\mathrm{d}^3x &&= -\frac{GMm}{r}\,\frac{1}{\gamma}\\
U_\mathrm{V} &= \int \phi^\mu\,\frac{j_\mu}{c}\,\mathrm{d}^3x &&= -\frac{GMm}{r}\\
U_\mathrm{T} &= \int \phi^{\mu\nu}\,\frac{K_{\mu\nu}}{c^2}\,\mathrm{d}^3x &&= -\frac{GMm}{r}\,\gamma\\
L &= -\frac{mc^2}{\gamma} - U && \text{expanded to order}
\end{aligned}$$

$$\beta^2 = \frac{v^2}{c^2} \approx \frac{r_0}{r} \qquad r_0 = \frac{GM}{c^2} = \tfrac{1}{2}\,r_{\text{Schwarzschild}}$$

$$\Omega_{\mathrm{V}} = \frac{\omega}{2}\,\frac{r_0}{a}\,\frac{1}{1-e^2} = \tfrac{1}{6}\,\Omega_{\mathrm{GR}} = 7.16''/\mathrm{cent}$$

$$\Omega_{\mathrm{S}} = -\Omega_{\mathrm{V}}$$

$$\Omega_{\mathrm{T}} = 3\,\Omega_{\mathrm{V}} = \tfrac{1}{2}\,\Omega_{\mathrm{GR}}$$

$$\Omega_{\mathrm{Spin}\,2} = \tfrac{7}{6}\,\Omega_{\mathrm{GR}} \quad \longleftarrow \quad 2\left(\phi^{\mu\nu} - \tfrac{1}{2}\,\eta^{\mu\nu}\phi\right)$$

$$= 2\,\Omega_{\mathrm{T}} - \Omega_{\mathrm{S}}$$

$$\boxed{\begin{aligned} \omega &= \frac{2\pi}{T} \\ T &= .2409\ \text{sidereal yrs} \\ a &= 5.79 \times 10^7\ \text{km} \\ r_0 &= 1.4766\ \text{km} \\ e &= .2056 \end{aligned}}$$

2 Feynman's approach

This lecture follows Feynman's conventions

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + 2\lambda h_{\mu\nu} \qquad \lambda \equiv \frac{\sqrt{8\pi G}}{c}} \tag{1}$$

instead of $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}/c^2$ and $\lambda = GMm$, except that Feynman puts $c = 1$. Indices are still raised and lowered with the flat space metric, and the bar operation is still

$$\overline{h}^{\mu\nu} \equiv h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\,h \qquad\qquad h \equiv h^{\sigma}{}_{\sigma} \qquad\qquad \overline{\overline{h}}^{\mu\nu} = h^{\mu\nu} \tag{2}$$

2.a Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\left(h_{\mu\nu,\lambda}\,\overline{h}^{\mu\nu,\lambda} - 2\,\overline{h}_{\mu\lambda,}{}^{\lambda}\,\overline{h}^{\mu\sigma,}{}_{\sigma}\right) - \lambda h_{\mu\nu}\,T^{\mu\nu} + \mathcal{L}_{\mathrm{matter}} \tag{3}$$

2.b Field equation

$$h_{\mu\nu,\sigma}{}^{\sigma} - \left(\overline{h}_{\mu\sigma,}{}^{\sigma}{}_{\nu} + \overline{h}_{\nu\sigma,}{}^{\sigma}{}_{\mu}\right) = -\lambda\,\overline{T}_{\mu\nu} \tag{4}$$

Then $\partial_\mu T^{\mu\nu} = 0$ is automatic if $\overline{h}^{\mu\nu}$ obeys the gauge condition. In these units, $h = O(\lambda)$.

2.c Matter equations

For a point mass:

$$T^{\mu\nu} = mc \int \delta[x - x(\alpha)]\, \dot{x}^{\mu}(\alpha)\, \dot{x}^{\mu}(\alpha)\, \mathrm{d}\alpha$$
$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}\alpha} \qquad \text{dim } \alpha = \text{time, } \dot{x}\cdot\dot{x} \text{ unconstrained} \tag{5}$$

$$\begin{aligned} S_{\text{matter}} + S_{\text{int}} &= \int \mathcal{L}_{\text{matter}} \frac{d^4x}{c} - \int \lambda h_{\mu\nu}\, T^{\mu\nu}\, \frac{d^4x}{c} \\ &= -\tfrac{1}{2} \int m\, \dot{x}\cdot\dot{x}\, \mathrm{d}\alpha - \lambda m \int h_{\mu\nu}\, \dot{x}^{\mu}\, \dot{x}^{\nu}\, \mathrm{d}\alpha \\ &\equiv \int \mathbb{L}\, \mathrm{d}\alpha \end{aligned} \tag{6}$$

E&M example

$$\mathbb{L} = -\tfrac{1}{2}\, m\, \dot{x}\cdot\dot{x} - \frac{q}{c}\, \dot{x}\cdot A \qquad \text{(unconstrained)} \tag{7a}$$

$$\Rightarrow \quad m\, \ddot{x}^{\mu} = \frac{q}{c}\, F^{\mu\nu}\, \dot{x}_{\nu} \tag{7b}$$

Then $\dot{x}\cdot\ddot{x} = 0$, because F is antisymmetric, $\Rightarrow$ $\dot{x}\cdot\dot{x}$ = const, $\Rightarrow$ $\alpha \propto$ proper time.

Equation of motion In our case

$$-\eta_{\mu\nu}\, \ddot{x}^{\nu} - 2\lambda \frac{\partial h_{\mu\nu}}{\partial x^{\lambda}}\, \dot{x}^{\lambda}\, \dot{x}^{\nu} + \lambda \frac{\partial h_{\lambda\nu}}{\partial x^{\mu}}\, \dot{x}^{\lambda}\, \dot{x}^{\nu} - 2\lambda\, h_{\mu\nu}\, \ddot{x}^{\nu} = 0 \tag{8}$$

Define

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + 2\lambda h_{\mu\nu} \tag{9}$$

and the usual Christoffel symbol with three downs

$$[\,\mu\nu,\, \sigma\,] \equiv \tfrac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \tag{10}$$

Then the equation of motion

$$\boxed{g_{\mu\nu}\,\ddot{x}^{\nu} = -[\,\nu\,\lambda,\,\mu\,]\,\dot{x}^{\nu}\,\dot{x}^{\lambda}} \tag{11}$$

implies the constraint

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\right) = 0 \qquad \Rightarrow \qquad g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} = c^2 \quad \text{(const)} \tag{12}$$

Now α is *not* proper time, but depends on the gravitational field. Feynman says the bending of light is correct, and the precession of Mercury is

$$\boxed{\Omega_{\mathrm{F}} = \tfrac{4}{3}\,\Omega_{\mathrm{GR}}} \tag{13}$$

3 Perturbed equation of motion

$$m\,\boldsymbol{a} = -\frac{GMm\,\boldsymbol{r}}{r^3} + \boldsymbol{f}_{\mathrm{pert}} \equiv \boldsymbol{f}_{\mathrm{Newt}} + \boldsymbol{f}_{\mathrm{pert}} \tag{14}$$

3.a Old (proper time formalism)

$$\frac{\boldsymbol{f}_{\mathrm{pert}}}{m} = \frac{\boldsymbol{f}_{\mathrm{Newt}}}{m}\left(\frac{h_1-1}{2}\,\beta^2 - h_1\,\frac{r_0}{r}\right) - \frac{\boldsymbol{f}_{\mathrm{Newt}}}{m}\cdot\frac{\boldsymbol{v}}{c}\,\frac{\boldsymbol{v}}{c}\,(1+h_1) \tag{15}$$

$$\boxed{\begin{array}{llll} h_1 = & -1\,, & 0\,, & \dfrac{5}{3} \\ & \mathrm{S} & \mathrm{V} & \mathrm{T}_{\text{sym-traceless}} \end{array}}$$

3.b New (gravitational time formalism)

$$\frac{\boldsymbol{f}_{\mathrm{pert}}}{m} = \frac{\boldsymbol{f}_{\mathrm{Newt}}}{m}\left(\beta^2 - 2\,\frac{r_0}{r}\right) - 4\,\frac{\boldsymbol{f}_{\mathrm{Newt}}}{m}\cdot\frac{\boldsymbol{v}}{c}\,\frac{\boldsymbol{v}}{c} \tag{16}$$

I have not checked the bending of light, which I found to be impossible in the proper time formalism. I have verified the precession result by my own methods (ordinary time, expansion of equations of motion, classical secular perturbation theory).

At this point, the gravitational field equation is the same as before (for spin 2), linear. But the particle moves effectively in a curved metric ($\alpha \neq$ proper time).

4 What's wrong?

For matter

$$g_{\mu\nu}\,\ddot{x}^{\nu} = -[\,\nu\,\lambda,\ \mu\,]\,\dot{x}^{\nu}\dot{x}^{\lambda} \tag{17}$$

$$T^{\mu\nu} = \int \delta[x - x(\alpha)]\, m\, \dot{x}^{\mu}(\alpha)\, \dot{x}^{\nu}(\alpha)\, c\mathrm{d}\alpha \tag{18}$$

implies that $T^{\mu\nu}$ obeys

$$\begin{aligned} g_{\mu\nu}\, T^{\nu\lambda}{}_{,\lambda} &= g_{\mu\nu}(x) \int \delta[x - x(\alpha)]\, m\, \ddot{x}^{\nu}(\alpha)\, c\mathrm{d}\alpha \\ &= -[\,\nu\,\lambda,\ \mu\,] \int \delta[x - x(\alpha)]\, m\, \dot{x}^{\nu}(\alpha)\, \dot{x}^{\lambda}(\alpha)\, c\mathrm{d}\alpha \end{aligned} \tag{19}$$

(note the shuffle between x and $x(\alpha)$), i.e.,

$$\boxed{\eta_{\mu\nu}\, T^{\nu\lambda}{}_{,\lambda} = -[\,\nu\,\lambda,\ \mu\,]\, T^{\nu\lambda} - 2\lambda h_{\mu\nu}\, T^{\nu\lambda}{}_{,\lambda}} \tag{20}$$

$\neq 0$! This is *exact*. $T^{\mu\nu}$ for a mass point obeys this equation of motion in the exact theory. We're going to change only the *field equation*.

The field equation is inconsistent with this because it follows from it that $\partial_{\mu} T^{\mu\nu} = 0$. We didn't get into trouble in our previous calculation because we had two different $T^{\mu\nu}$'s, $T^{\mu\nu}_{\text{sun}}$ and $T^{\mu\nu}_{\text{Mercury}}$, and $\partial_{\mu} T^{\mu\nu}_{\text{sun}} = 0$ because the sun is at rest, while we never wrote a field equation with $T^{\mu\nu}_{\text{Mercury}}$ as a source.

5 Feynman's strategy

Add a term to $\mathcal{L}_{\text{grav}}$ such that

(i) The field equation becomes

$$h_{\mu\nu,\sigma}{}^{\sigma} - \left(\overline{h}_{\mu\sigma,}{}^{\sigma}{}_{\nu} + \overline{h}_{\nu\sigma,}{}^{\sigma}{}_{\mu}\right) = -\lambda\left(\overline{T}^{\mu\nu} + \overline{\chi}^{\mu\nu}\right) \quad \text{(all indices flat)} \tag{21}$$

Then the equation is forced:

$$\boxed{T^{\mu\nu}{}_{,\nu} + \chi^{\mu\nu}{}_{,\nu} = 0 \quad \text{(flat)}} \tag{22}$$

Here $T^{\mu\nu}$ is stil the matter tensor, and $\chi^{\mu\nu}$ is interpreted as the gravitational (self-energy) contribution.

So far the solution for the functional form of the extra Lagrangian could be *zero*.

(ii) The matter equation (17) constrains the extra piece of the Lagrangian through (20). Putting that into the above gives

$$\boxed{\eta_{\mu\nu}\,\chi^{\nu\lambda}{}_{,\lambda} = [\,\nu\,\lambda,\,\mu\,]\,T^{\nu\lambda} + 2\lambda h_{\mu\nu}\,T^{\nu\lambda}{}_{,\lambda}} \tag{23}$$

Since $[\,..,\,.\,]$ is linear in λh and thus $[O(\lambda^2)]$, the leading order on the r.h.s. is

$$\eta_{\mu\nu}\,\chi^{\nu\lambda}{}_{,\lambda} = [\,\nu\,\lambda,\,\mu\,]\,T^{\nu\lambda} + O(\lambda^4) \tag{24}$$

(Replace $T^{\nu\lambda}{}_{,\lambda}$ by $-\chi^{\nu\lambda}{}_{,\lambda}$.)

I guess the idea now is to replace $T^{\nu\lambda}$ on the r.h.s. by the lowest order field equation. (*Yep!*) Feynman claims the result for χ is the variation of a function $F^3 = \lambda \times$ (trilinear in h).[1] He claims the precession is now correct as far as observations go. He does not say whether $\Omega = \Omega_{\rm GR}$ exactly (presumably it is not), and I haven't checked this.

6 Theory correct to all orders

This procedure could be iterated, but the algebra is already horrendous at the level of the first correction. Feynman "guesses" *a* solution of the exact problem. Choose $\mathcal{L}_{\rm grav}$ such that the resulting field equation enforces automatically:[2]

$$\boxed{g_{\mu\nu}\,T^{\nu\lambda}{}_{,\lambda} = -[\,\nu\,\lambda,\,\mu\,]\,T^{\nu\lambda}} \tag{25}$$

This is of course the vanishing of the covariant divergence from a flat space viewpoint.

[1] Intially containing 18 forms with 18 constants. Feynman, *op. cit.*, p. 75.

[2] Do not change $\mathcal{L}_{\rm int} + \mathcal{L}_{\rm matter}$. Leave them the same as (22) and (23).

6.a Lagrangian

The claim is that G.R. is a solution. After the definitions

$$g^{\mu\nu}\,g_{\nu\lambda} = \delta^{\mu}{}_{\lambda} \tag{26}$$

$$\left\{ {\tau \atop \mu\,\nu} \right\} = g^{\tau\sigma}\,[\,\mu\,\nu,\,\sigma\,] \tag{27}$$

$$R^{\tau}{}_{\mu\nu\rho} = \left\{ {\tau \atop \mu\,\nu} \right\}_{,\rho} + \left\{ {\tau \atop \rho\,\lambda} \right\}\left\{ {\lambda \atop \mu\,\nu} \right\} - \left\{ {\tau \atop \mu\,\rho} \right\}_{,\nu} - \left\{ {\tau \atop \nu\,\lambda} \right\}\left\{ {\lambda \atop \mu\,\rho} \right\} \tag{28}$$

the result is

$$\boxed{\mathcal{L}_{\text{grav}} = -\frac{1}{2\lambda^2}\,g^{\mu\nu}\,R_{\mu\nu}\,\sqrt{-\det g_{\lambda\rho}} \qquad R_{\mu\nu} \equiv R^{\tau}{}_{\mu\nu\tau}} \tag{29}$$

where presumably integration by parts is assumed to remove second derivatives.

6.b Field equation

With the same matter tensor $T^{\mu\nu}$ as before

$$\boxed{\sqrt{-g}\,\left(R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\,R\right) = \lambda^2\,T^{\mu\nu}} \tag{30}$$

$$\Rightarrow$$

$$\boxed{g_{\mu\nu}\,T^{\nu\lambda}{}_{,\lambda} = -[\,\nu\,\lambda,\,\mu\,]\,T^{\nu\lambda}} \tag{31}$$

6.c Matter equations

$$\boxed{g_{\mu\nu}\,\ddot{x}^{\nu} = -[\,\nu\,\lambda,\,\mu\,]\,\dot{x}^{\nu}\,\dot{x}^{\lambda}} \tag{32}$$

$$\Rightarrow$$

$$\boxed{g_{\mu\nu}\,\dot{x}^{\nu}\,\dot{x}^{\nu} = c^2} \tag{33}$$