On Stapp's Theorem*

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I. Introduction

The purpose of this paper is to clarify a few points concerning the proof of an important theorem of H. P. Stapp [1,2] on holomorphic, Lorentz-covariant functions. Stapp's Theorem is a generalization to arbitrary domains, under somewhat weaker conditions, of a result first proved by D. Hall and A. S. Wightman [3] (invariant functions), and generalized by R. Jost [4] (covariant functions), for the "future tube". It says essentially that the analytic continuation of a holomorphic function of four-vectors, covariant under the real, connected Lorentz group, is covariant under the complex, connected Lorentz group.

In order to describe the theorem more precisely, we must introduce some terminology. Denote the connected part of the real, homogeneous Lorentz group by $\mathrm{L}_+^\uparrow$, and the proper, complex, homogeneous Lorentz group by $\mathcal{L}_+$. Specifically, $\mathcal{L}_+$ is the set of unimodular, complex, 4×4 matrices Λ satisfying

$$\Lambda G \Lambda^{\mathrm{T}} = G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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where Λ^{T} means the transpose of Λ. We consider domains (connected, open sets) consisting of points Z in the complex number space $\mathbb{C}^{4\ell}$, (or occasionally in the real number space $\mathbb{R}^{4\ell}$) $Z = (z_1, \ldots, z_\ell)$, where each z_i is a four-vector. For $\Lambda \in \mathcal{L}_+$ we write $\Lambda Z = (\Lambda z_1, \ldots, \Lambda z_\ell)$. We say that Z and $Z' = \Lambda Z$, with $\Lambda \in \mathcal{L}_+$, are $\mathcal{L}_+$-*equivalent*.

For an mth rank tensor $f^{\mu_1 \cdots \mu_m}$, with $\mu_i = 0, 1, 2, 3$, we use the short notation

$$\Lambda^{\mu_1}{}_{\nu_1} \ldots \Lambda^{\mu_m}{}_{\nu_m} f^{\nu_1 \cdots \nu_m} = (\Lambda f)^{\mu_1 \cdots \mu_m},$$

most often without writing out the indices.

Definition 1. An $\mathrm{L}_+^\uparrow$ (or $\mathcal{L}_+$)-covariant function f on a set $A \subset \mathbb{C}^{4\ell}$ is a tensor-valued function on A satisfying the equation

$$f(Z) = \Lambda^{-1} f(\Lambda Z)$$

whenever Z and ΛZ are in A and Λ is in $\mathrm{L}_+^\uparrow$ (or $\mathcal{L}_+$).

It is sufficient to discuss tensor-valued and not spinor-valued functions because of the well-known fact that spinor-valued functions satisfying an equation like the above must vanish.

We use without much explanation various well-known concepts from the theory of holomorphic functions of several complex variables [5]. For exmple, a *locally schlicht domain* over $\mathbb{C}^{4\ell}$ (or over the complex mass shell) is a pair (S, ϕ), where S is a Hausdorff space and ϕ a local homeomorphism of S into $\mathbb{C}^{4\ell}$ (or into the complex mass shell). The term *domain of regularity*, denoted by R, is used (as by Stapp [1]) for the "intersection" of the Riemann "domains of holomorphy" of the components of a holomorphic, tensor-valued function. The domain of holomorphy is just the space of equivalent function elements, a locally schlicht domain. A *sheet* is a schlicht subdomain of R not properly contained in any other schlicht subdomain of R. We use the same symbol for f and its analytic continuation onto R.

For simplicity, we discuss domains in or over $\mathbb{C}^{4\ell}$, or more generally, $\mathbb{C}^{n\ell}$, but all results carry over quite easily to holomorphic functions on domains over the complex mass shell [1, 6]. When we are discussing schlicht domains $U \subset R$, we make no distinction between a point $P \in U$ and its projection $\phi(P) = Z \in \mathbb{C}^{n\ell}$.

Stapp's basic result is

Stapp's Theorem. Let f be $\mathrm{L}_+^\uparrow$-covariant on a schlicht domain $D \subset \mathbb{R}^{4\ell}$, and holomorphic at each point of D. Let R be the domain of regularity of f, over $\mathbb{C}^{4\ell}$. Then R is a union of sheets (which may overlap), each of which maps onto itself under every transformation in $\mathcal{L}_+$, and on each of which f is $\mathcal{L}_+$-covariant. If in addition D is such that any two points related by a transformation in $\mathcal{L}_+$ are also related by a transformation in $\mathrm{L}_+^\uparrow$, then every compact subset of D is contained in a single $\mathcal{L}_+$-invariant sheet, on which f is $\mathcal{L}_+$-covariant. (This condition is satisfied when D consists of physical points on the mass shell.)

In other words, holomorphy and real covariance on a real domain, no matter how small, imply complex covariance for the entire domain of regularity. There are variations to this theorem which are not of much concern to us here. For example, if D is

replaced by a complex domain, and $L_+^\uparrow$ covariance on D by $\mathcal{L}_+$ covariance on D, one still gets a true theorem [1].

For comparison, recall that in Hall and Wightman's [3] Lemma 1, $L_+^\uparrow$ invariance and holomorphy on the future tube imply $\mathcal{L}_+$ invariance and the existence of an analytic continuation onto the extended future tube.

The keystone of Stapp's proof is a local property of $\mathcal{L}_+$ orbits. The $\mathcal{L}_+$ *orbit* corresponding to a point Z is the set of points

$$\mathcal{L}_+ Z = \{ Z' : Z' = \Lambda Z \, , \Lambda \in \mathcal{L}_+ \}.$$

We also use the notion of an $\mathcal{L}_+$ orbit in a locally schlicht domain, such as R. Then it just means the set of points obtained from a given point by continuation of the corresponding function element along the images of all possible paths in $\mathcal{L}_+$, beginning at the identity.

Definition 2. We say that a set A is $\mathcal{L}_+$-*connected* in a set $B \subset \mathbb{C}^{4\ell}$ if $A \subset B$ and if for any pair of $\mathcal{L}_+$-equivalent points, Z and Z' in A, there is an arc (the topological image of a closed interval) $\Lambda(t) \in \mathcal{L}_+$, for $0 \leq t \leq 1$, with $\Lambda(0) = I$ and $\Lambda(1)Z = Z'$, such that $\Lambda(t)Z \in B$.

The property in question is

Local $\mathcal{L}_+$ Connectedness. The space $\mathbb{C}^{4\ell}$ is said to be *locally* $\mathcal{L}_+$-*connected* at a point Z if for every neighborhood U of Z there is a neighborhood U_0 of Z that is $\mathcal{L}_+$-connected in U.

The difficult part of the proof of Hall and Wightman's Lemma 1 was just to show that the future tube is $\mathcal{L}_+$-connected in itself.

Stapp's Lemma 4 says that $\mathbb{C}^{4\ell}$ is everywhere locally $\mathcal{L}_+$-connected, but this statement has recently turned out to be too strong. R. Jost has constructed an ingenious counterexample. The counterexample applies only at certain pathological points; and as Jost has pointed out [7], it has little practical effect on Stapp's main result. However, as one of us (RS) will indicate in Section IV, it completely characterizes the points where the property does not hold; and with Stapp's rather elegant proof for the "good" points (which are dense in $\mathbb{C}^{4\ell}$), it goes a long way towards bringing about a definitive understanding of the local properties of $\mathcal{L}_+$ orbits.

With the pathological points taken into account, Stapp has proved the theorem for the I_+-*saturated kernel* [6] of the domain of regularity, R, plus any additional points with Grammian,

$$G(Z) = \det \left(z_i \cdot z_j \right),$$

of rank two. The I_+-saturated kernel is defined as follows: Let$P \in R' \subset R$, and let the projection of P in $\mathbb{C}^{4\ell}$ be $Z = \phi(P)$. The set of $\mathcal{L}_+$ *invariants* of the point Z is the set of scalar and pseudoscalar invariants, $z_i \cdot z_j$ and

$$\epsilon_{\mu\nu\lambda\rho} \, z_{i_1}^\mu \, z_{i_2}^\nu \, z_{i_3}^\lambda \, z_{i_4}^\rho \, ,$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the alternating symbol. Now R' is said to be I_+-saturated if, for every $P \in R'$, there is a schlicht domain U containing P such that the image, $\phi(U \cap R') \subset \mathbb{C}^{4\ell}$, contains all points with the same $\mathcal{L}_+$ invariants as the point $Z = \phi(P)$. R' is the I_+-saturated kernel of R if it is the largest I_+-saturated subset of R. Then R' is a dense subdomain of R [6]. Because it is characterized by invariants, it is to be regarded as the physically interesting part of R.

Thus, Stapp has proved a substantial result, which gives all that one can in practice demand.

By a different method, using a decomposition in "standard covariants" and a continuation in the Riemann domain over the variety of $\mathcal{L}_+$ invariants, K. Hepp [6] has also proved a result similar to Stapp's, for the I_+-saturated kernel of R, starting out from covariance under the complex group in some domain.[1]

But even though it is perhaps only of mathematical interest, there remains the question: Is Stapp's Theorem true for the entire domain of regularity? In this paper, the first two authors present two proofs that it is. We do this by proving one of Stapp's lemmas (Lemma 8) on local $\mathcal{L}_+$ covariance:

Theorem A. Let f be $\mathrm{L}_+^\uparrow$-covariant and holomorphic on a real domain $D \subset \mathbb{R}^{4\ell}$. Let R be the Riemann domain of regularity of f, over $\mathbb{C}^{4\ell}$. Then for every $P \in R$ and every neighborhood U of P contained in R there is a neighborhood U_0 of P, with $U_0 \subset U$, such that f is $\mathcal{L}_+$-covariant on U_0.

(Just as in Stapp's Theorem, one can replace the real domain D by a complex domain, and "$\mathrm{L}_+^\uparrow$-covariant" by "$\mathcal{L}_+$-covariant", and still get a true result. And as already mentioned, everything goes through when $\mathbb{C}^{4\ell}$ is replaced by the complex mass shell.) Stapp's only essential use of local $\mathcal{L}_+$ connectedness is in the proof of Theorem A; he makes no further use of that property in deriving the invariant sheet structure of R. Our proofs of Theorem A are independent of local $\mathcal{L}_+$ connectedness.

Section II introduces a few preliminary concepts, including some essential results of Stapp. We give the discussion a quite general form, with conditions that suffice to prove the theorem for a large class of connected, complex Lie groups of linear transformations on spaces of any finite dimension. The group $\mathcal{L}_+$ in particular satisfies the conditions. This of course inverts the order in which the proofs were originally developed, but it has the advantage of making clear the points where the specific structure of the group enters.

Section III contains proofs of a generalized form of Theorem A, due to DW and PM. The version given here is the result of some refinement, which made it possible to collapse the two proofs into one, except for the final steps of each.

In Section IV, local $\mathcal{L}_+$ connectedness is discussed by RS. Although, as already mentioned, Stapp's proof is sufficient for the points where the property holds, a different proof is given here, for the sake of completeness.

In the Conclusion, we remark on the extent to which our results are valid, or can be expected to be valid, for the complex, classical groups of linear transformations.

[1] A. O. Barut has given a plausibility argument, based on ideas similar to those of Hepp. See *Strong Interactions and High Energy Physics*, Oliver and Boyd, Ltd., Edinburgh, 1964, R. G. Moorhouse, ed., pp. 94-95.

We want to emphasize that the relation of this paper to Stapp's work is that of clarifying some points that are sticky, but which have little practical effect on the outcome. In the sense that our proofs provide a simple, unified treatment for the whole domain of regularity, they are an improvement. This improvement naturally relies heavily on earlier results and methods of Stapp, and it certainly does not detract from the importance of his original work.

It is a pleasure to acknowledge our indebtedness to Professor Jost for permission to describe his counterexample, the study of which led to our proofs, as well as for his encouragement and suggestions, which either directly or indirectly got us onto the right track. We wish to thank Dr. K. Hepp for useful conversations, and one of us (DW) wishes to thank Professor M. Fierz for his hospitality during that author's stay at the ETH.

II. Basic Ingredients for the Proof

Let $Z = (z_1, \dots, z_\ell) \in \mathbb{C}^{n\ell}$, where each z_i is an n-vector from $\mathbb{C}^n$. We introduce the usual topology in $\mathbb{C}^{n\ell}$, along with some nonsingular unitary metric that generates it.

Let G(n, C) denote a connected, complex [8] Lie group of linear transformations of the space $\mathbb{C}^n$, and let G(n, R) denote a connected, real Lie group of linear tranformations of the real space $\mathbb{R}^n$. Every G(n, R) is a subgroup of a particular G(n, C) which is its *analytic complexification*. This means that there is an analytic parameterization of a neighborhood of the identity in G(n, C) that also parameterizes a neighborhood of the identity in G(n, R), when restricted to real values [8].

For any Λ in G(n, C) or G(n, R), write $\Lambda Z = (\Lambda z_1, \dots, \Lambda z_\ell)$. By obvious analogy with the definitions for $L_+^\uparrow$ and $\mathcal{L}_+$ in Section I, we define the G(n, C) orbit of a point Z, the G(n, C) equivalence of two points of $\mathbb{C}^{n\ell}$, and the G(n, R) or G(n, C) covariance of tensor-valued functions of Z.

The first requirement for our proof is a kind of *weak G(n, C) covariance*, a property that is both described in and guaranteed by the lemma below. This result is already implicit in certain of Stapp's lemmas, although a small additional argument, supplied in the Appendix, is needed.

Lemma A. Let f be G(n, C)-covariant and holomorphic on a domain $D \subset \mathbb{C}^{n\ell}$, and let R be the Riemann domain of regularity, over $\mathbb{C}^{n\ell}$. Alternatively, let f be only G(n, R)-covariant and holomorphic on a real domain $D \subset \mathbb{R}^{n\ell}$, and let G(n, C) be the analytic complexification of G(n, R).

Then f is *weakly G(n, C)-covariant* on R. That is:

(i) For every $P \in R$ and every arc, $\Lambda(t) \in$ G(n, C), with $0 \le t \le 1$ and $\Lambda(0) = I$, it follows that $\Lambda(t)P \in R$ and that

$$\Lambda(t) f(P) = f[\Lambda(t)P]\,;$$

(ii) Let U be a domain, with compact closure contained in a schlicht subdomain U' of R, that is, $U \subset\subset U' \subset R$. For $\Lambda \in$ G(n, C), define ΛU by means of some arc $\Lambda(t) \in$ G(n, C), with $0 \le t \le 1$, $\Lambda(0) = I$, and $\Lambda(1) = \Lambda$. Then ΛU is a schlicht subdomain of R.

K. Hepp has pointed out [7] that, for $\mathcal{L}_+$, part (i) of this lemma is also a consequence of Stapp's Theorem for the I_+-saturated kernel of R (or even for the domain of points of R whose Grammian has maximum rank), by continuity arguments. The proof in the Appendix gives (i) and (ii) simultaneously.

Note that although Lemma A says that f is G(n, C)-covariant along paths $\Lambda(t)P$, it does not say that f is G(n, C)-covariant for G(n, C)-equivalent points, P and P', of a small, schlicht neighborhood. It is true that there is a path $\Lambda(t)P$ that connects P to a point of R that at least *lies over* P', but *a priori* it may lie on a different sheet. If something like the local $\mathcal{L}_+$ connectedness property holds, then the G(n, C) covariance of f for P and P', and hence the generalization of Theorem A, is immediate from Lemma A. The point of our discussion is that the same conclusion holds without "local G(n, C) connectedness".

We do, however, require that the group G(n, C) have two other properties, the first of which amounts to a weak form of local G(n, C) connectedness.

Condition I. Let $Z \in \mathbb{C}^{nd}$, with $d \leq n$, and let $z_1, \ldots, z_d$ be linearly independent, *i.e.*, Dim $Z = d$. Then for every neighborhood of the identity, $\mathcal{N}(I) \subset$ G(n, C), there is a neighborhood $U(Z) \subset \mathbb{C}^{nd}$ such that, if Z' and Z'' are G(n, C)-equivalent in U, then there is a $\Lambda \in \mathcal{N}(I)$ satisfying $Z'' = \Lambda Z$.

For $\mathcal{L}_+$, this nontrivial condition is Stapp's Lemma 2.

Secondly, we need a property of the *little group*, G(n, C : Z), of a point $Z \in \mathbb{C}^{n\ell}$. We define

$$\mathrm{G(n, C : } Z) = \{\Lambda \in \mathrm{G(n, C)} : \Lambda Z = Z\}\,.$$

Condition II. For every $Z \in \mathbb{C}^{n\ell}$, the little group G(n, C : Z) is connected.

The proof that Condition II holds for $\mathcal{L}_+$ is straightforward, by an enumeration of cases. In those cases where Dim Z = Rank $G(Z)$, $\mathcal{L}_+(Z)$ is isomorphic and topologically equivalent to one of the complex, proper, orthogonal groups, $\mathrm{O}_+(n, \mathrm{C})$, for $n = 1, 2, 3, 4$, which are connected. The remaining few cases are easily parameterized, giving the result by inspection.

If we now take Conditions I and II for granted, the remaining parts of the proofs are quite simple. We reserve for the Conclusion a discussion of the extent to which these conditions hold, or can be expected to hold, for classical, complex groups other than $\mathcal{L}_+$.

III. Proof of the General Theorem

We shall prove the following:

Theorem B. Let f be a holomorphic, tensor-valued function, with Riemann domain of regularity R over $\mathbb{C}^{n\ell}$. Let f be G(n, C)-covariant on some schlicht subdomain of R; or let f be only G(n, R)-covariant on some schlicht subset of R that is a domain over $\mathbb{R}^{n\ell}$, and let G(n, C) be the analytic complexification of G(n, R). Let G(n, C) satisfy Conditions I and II.

Then for every $P \in R$ and every neighborhood $U(P) \subset R$, there is a neighborhood $U_0(Z) \subset U(Z)$ such that f is G(n, C)-covariant on $U_0(Z)$.

As we have seen, $\mathcal{L}_+$ satisfies the conditions, so that Theorem A in Section I is a special case of Theorem B.

Proof.

1. Note that the conditions of Lemma A are satisfied. Thus f is weakly G(n, C) covariant on R.

2. To simplify the proof [9], we first make a linear transformation of the space R, depending only on the fixed point P, by means of an $\ell \times \ell$, nonsingular matrix A, acting on the indices $i = 1, \dots, \ell$ that label the vectors corresponding to points of R. It is clear that any such transformation is holomorphic and topological, and that it preserves the relation of G(n, C) equivalence, as well as G(n, C) orbits, so that the argument is not affected. The fact that R is not schlicht causes no trouble; just carry out the transformation on each function element.

Choose A as follows: Let Dim $P = d$. Let A be such that

$$q_i = \sum_{j=1}^{\ell} P_j A_{ji} \qquad i = 1, \dots, d\,,$$

$$0 = \sum_{j=1}^{\ell} P_j A_{ji} \qquad i = d+1, \dots, \ell\,,$$

where the q_i are a set of linearly independent n-vectors.

Now consider the space $\mathbb{C}^{n\ell}$ as a Cartesian product, $\mathbb{C}^{nd} \times \mathbb{C}^{n(\ell-d)}$, with the usual product topology, and write the points $Z \in \mathbb{C}^{n\ell}$ as ordered pairs

$$Z = (\xi, \eta)\,, \qquad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{C}^{nd},$$
$$\eta = (\eta_1, \dots, \eta_{\ell-d}) \in \mathbb{C}^{n(\ell-d)}.$$

Thus, if we choose A as above, we need only consider neighborhoods of a fixed point of the form

$$\alpha = (\beta, 0),$$

which we can take to be schlicht, Cartesian product domains of the form

$$U(\alpha) = V(\beta) \times W(0),$$

where $V(\beta) \subset \mathbb{C}^{nd}$ and $W(0) \subset \mathbb{C}^{n(\ell-d)}$.

The next step is essentially to reduce the whole problem to a consideration of neighborhoods $W(0)$ of $0 \in \mathbb{C}^{n(\ell-d)}$.

3. Let $U(\alpha) \subset R$ be a schlicht neighborhood of $\alpha = (\beta, 0)$. Choose a neighborhood $U_1(\alpha) = V_1(\beta) \times W_1(0) \subset\subset U(\alpha)$, with $W_1(0)$ convex. Then choose a connected, open neighborhood of the identity, $\mathcal{N}(I) \subset \mathrm{G(n, C)}$, such that $\Lambda U_1(\alpha) \subset U(\alpha)$ for all $\Lambda \in \mathcal{N}(I)$. Finally, choose a neighborhood $V_0(\beta) \subset V_1(\beta)$ such that $V_0(\beta)$ satisfies Condition I for $\mathcal{N}(I)$.

We shall prove the G(n, C) covariance of f on the domain $U_0(\alpha) = V_0(\beta) \times W_1(0)$.

Let $\zeta = (\xi, \eta)$ and $\zeta'' = (\xi'', \eta'')$ be G(n, C)-equivalent points of $U_0(\alpha)$. It follows from Condition I that there is a $\Lambda_1 \in \mathcal{N}(I)$ with $\xi = \Lambda_1 \xi''$, and from the connectedness of $\mathcal{N}(I)$ that there is an arc $\Lambda_1(t)$, with $0 \leq t \leq 1$, $\Lambda_1(0) = I$, and $\Lambda_1(1) = \Lambda_1$, such that $\Lambda_1(t)\xi'' \in U_1(\alpha)$.

Let $\zeta' = \Lambda_1 \zeta'' = (\xi, \eta') \in U_1(\alpha)$. Then the weak G(n, C) covariance of f implies that

$$f(\zeta'') = \Lambda_1^{-1} f(\zeta'),$$

and thus we need only consider G(n, C)-equivalent points ζ and ζ' in $U_1(\alpha)$ with

$$\zeta = (\xi, \eta), \qquad \zeta' = (\xi, \eta').$$

In other words, we need in effect only consider $\mathrm{G(n, C : \xi)}$-equivalent points η and η' in $W_1(0)$, where $\mathrm{G(n, C : \xi)}$ is the little group of ξ and $\xi \in V_1(\beta)$.

4. Both of the proofs that follow make use of the following construction. Let $\zeta = (\xi, \eta)$ and $\zeta' = (\xi, \eta')$ be G(n, C)-equivalent points of $U_1(\alpha)$, with $\zeta' = \Lambda\zeta$. Consider sets of points parameterized by a complex number λ:

$$\zeta(\lambda) = (\xi, \lambda\eta), \qquad \zeta'(\lambda) = (\xi, \lambda\eta').$$

These points have the following properties:

(i) For some neighborhood of the real segment $0 \leq t \leq 1$, the convexity of $W_1(0)$ implies that $\zeta(\lambda) \in U_1(\alpha)$ and $\zeta'(\lambda) \in U_1(\alpha)$.

(ii) $\zeta(0) = \zeta'(0) = \Lambda\zeta(0)$; $\zeta(1) = \zeta$; $\zeta'(1) = \zeta'$.

(iii) For all λ, $\Lambda\zeta(\lambda) = \zeta'(\lambda)$.

Finally, note that because $\mathrm{G(n, C : \xi)}$ is connected (Condition II), there is an arc $\Lambda(t) \in \mathrm{G(n, C : \xi)}$, with $0 \leq t \leq 1$, $\Lambda(0) = I$, and $\Lambda(1) = \Lambda$.

5. From this point, the proof will be completed in two different ways.

(i) Use the weak G(n, C) covariance of f (part (ii) of Lemma A) and the arc $\Lambda(t) \in \mathrm{G(n, C : \xi)}$ above to define the schlicht subdomain of R, $\Lambda U_1(\alpha)$. Then $\Lambda U_1(\alpha)$ contains a schlicht subdomain N that lies over a subdomain of $U_1(\alpha)$ containing the arc $\zeta'(\lambda)$, $0 \leq \lambda \leq 1$, because of (iii) above. But the point $\zeta(0) = \zeta'(0)$ satisfies $\Lambda(t)\zeta(0) = \zeta(0)$, because $\Lambda(t) \in \mathrm{G(n, C : \xi)}$. Thus $\zeta'(0)$ is common to N and $U_1(\alpha)$, and it therefore follows that $N \subset U_1(\alpha)$. (To put it another way, the continuation from the common point $\zeta'(0)$ along $\zeta'(\lambda)$ is unique.) The theorem is now immediate from weak G(n, C) covariance, *i.e.*, covariance along the orbit $\Lambda(t)\zeta$. □ (DW)

(ii) Consider the induced functions

$$\varphi(\lambda) = f(\xi, \lambda\eta) = f[\zeta(\lambda)];$$
$$\varphi'(\lambda) = f[\zeta'(\lambda)].$$

From the definitions of $\zeta(\lambda)$ and $\zeta'(\lambda)$, and the property (i) in §4, φ and φ' are analytic functions of λ for some neighborhood of the segment $0 \le \lambda \le 1$.

Now consider the points

$$\zeta(t, \lambda) = \Lambda(t)\zeta(\lambda),$$

with $\Lambda(t)$ defined as above. Because $\Lambda(t)$ is continuous on the compact set $0 \le t \le 1$, it is bounded; and thus, because $\Lambda(t)$ is a linear transformation in $\mathrm{G(n, C : \xi)}$, there is a number λ_1, $0 < \lambda_1 \le 1$, such that

$$\xi(t, \lambda) \in U_1(\alpha) \quad \text{for} \quad 0 \le t \le 1 \quad \text{and} \quad \lambda < \lambda_1.$$

From the weak G(n, C) covariance of f, *i.e.*, covariance along $\zeta(t, \lambda)$ for $\lambda < \lambda_1$, and from the analyticity of φ and φ' for $0 \le \lambda \le 1$, it follows that

$$\varphi'(1) = \Lambda\varphi(1), \quad \text{or} \quad f(\zeta') = \Lambda f(\zeta). \qquad \square\,\text{(PM)}$$

IV. Local $\mathcal{L}_+$ Connectedness

Although we have seen in the preceding sections that Stapp's Theorem depends only on a weak form of $\mathcal{L}_+$ connectedness, embodied in Condition I, it may be important in other applications to know the local geometrical properties of $\mathcal{L}_+$ orbits in more detail. The lemma below characterizes those points $Z \in \mathbb{C}^{4\ell}$ at which $\mathbb{C}^{4\ell}$ is locally $\mathcal{L}_+$-connected, that is, such that any neighborhood U' of Z contains a neighborhood U of Z that is $\mathcal{L}_+$-connected in U'. Here we are more interested in the points where the property does *not* hold; but for completeness, we discuss all points.

First, it may help to recall that the following relation exists between the dimension, Dim Z, of the space $\langle Z \rangle = \langle z_1, \ldots, z_\ell \rangle$ spanned by the four-vectors z_i of Z, and the rank of the Grammian, $G(Z)$:

Dim Z	Rank $G(Z)$
4	4
3	3, 2
2	2, 1, 0
1	1, 0
0	0

Lemma B (local $\mathcal{L}_+$ connectedness)**.** The space $\mathbb{C}^{4\ell}$ is locally $\mathcal{L}_+$-connected at Z if and only if Z satisfies one of the following conditions:

Dim Z = 4, 3, or 0;
Dim Z = Rank $G(Z)$ = 1 or 2.

Note that Lemma B implies that $\mathbb{C}^{4\ell}$ is locally $\mathcal{L}_+$-connected for a dense subdomain of points, for example, if $\ell \geq 4$, the set of points Z with Dim $Z = 4$, and if $\ell < 4$, the set of points with Dim $Z = \ell =$ Rank $G(Z)$.

In the proof, we use the following notation: Let U and V be orthogonal subspaces of $\mathbb{C}^4$ with respect to the Lorentz metric. Then we write $U \perp V$ for the orthogonal sum of U and V if and only if $U \cap V$ contains only the zero vector.

If $\mathbb{C}^4$ can be decomposed as the orthogonal sum, $\mathbb{C}^4 = U \perp V$, and if U is invariant under the transformation $\Sigma \in \mathcal{L}_+$, we use the notation $\Sigma = \Sigma_1 + \Sigma_2$, where $\Sigma_1 = \Sigma/U$ and $\Sigma_2 = \Sigma/V$ are, respectively, the transformations that are Σ on the spaces U and V and zero on the spaces V and U.

For neighborhoods of radius ε, we write

$$U(Z_0, \varepsilon) = \left\{ Z \in \mathbb{C}^{4\ell} : \|Z - Z_0\| < \varepsilon \right\},$$

where $\|Z\|$ is some norm corresponding to a unitary metric in $\mathbb{C}^{4\ell}$.

We use the convention that the first d vectors of Z are linearly independent, *i.e.*, Dim $\langle z_1, \ldots, z_d \rangle = d$.

Proof.

1. Dim $Z = 3$ or 4 Local $\mathcal{L}_+$ connectedness follows directly from Condition I in Section II, which is satisfied by $\mathcal{L}_+$ (Stapp's Lemma 2).

2. Dim $Z = 0$ We use a variation on Jost's normal form for a complex Lorentz transformation [10]. For every $\Lambda \in \mathcal{L}_+$ there is a decomposition:

$$\Lambda = U_1 \tilde{N}(\chi_1, \chi_2) U_2,$$

where U_1 and U_2 are unitary elements of $\mathcal{L}_+$ and

$$\tilde{N}(\chi_1, \chi_2) = T N(\chi_1, \chi_2) T^{-1}; \qquad T^* = T;$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}; \qquad N(\chi_1, \chi_2) = \begin{pmatrix} M(\chi_1) & 0 \\ 0 & M(\chi_2) \end{pmatrix};$$

$$M(\chi) = \begin{pmatrix} \cosh \chi & -i \sinh \chi \\ i \sinh \chi & \cosh \chi \end{pmatrix}.$$

It is enough to consider points K and K' in $U(Z, \varepsilon) \subset U'(Z)$ of the form $K' = \tilde{N}(\chi_1, \chi_2) K$. Then K and K' are connected by the path

$$K(t) = \tilde{N}(\chi_1 t, \chi_2 t) K, \qquad 0 \leq t \leq 1.$$

Because

$$\|K(0)\| < \varepsilon, \qquad \|K(1)\| < \varepsilon, \qquad \frac{d^2}{dt^2} \|K(t)\| \geq 0,$$

it follows that $K(t) \in U(Z, \varepsilon)$ for $0 \leq t \leq 1$.

3. Dim $Z = 2$

(i) Rank $G(Z) = 2$ We choose $U(Z, \varepsilon) \subset U'(Z)$ such that for all $K \in U(Z, \varepsilon)$ we have Rank $G(K) \geq 2$.

According to Condition I of Section II, it is enough to prove the statement for K and K' in $U(Z, \varepsilon)$ with $k_1 = k'_1$, $k_2 = k'_2$, and $K' = \Lambda K$. Then Λ is in the little group of (k_1, k_2), $\mathcal{L}_+(k_1, k_2)$, and has the decomposition

$$\Lambda = \Lambda / \langle k_1, k_2 \rangle + \Lambda / \langle k_1, k_2 \rangle^\perp$$

where $\mathbb{C}^4 = \langle k_1, k_2 \rangle \perp \langle k_1, k_2 \rangle^\perp$. Let

$$K^\perp = (0, 0, k_3^\perp, \ldots, k_\ell^\perp) \quad \text{and} \quad K'^\perp = (0, 0, k'^\perp_3, \ldots, k'^\perp_\ell)$$

where $k_i^\perp$ and $k'^\perp_i$ are the parts of k_i and k'_i in $\langle k_1, k_2 \rangle^\perp$. Then $\|K^\perp\| < \text{const} \times \varepsilon$ and $\|K'^\perp\| < \text{const} \times \varepsilon$, so that the problem is effectively reduced to the case Dim $Z = 0$ in 2ℓ dimensions.

(ii) Rank $G(Z) = 1$ For this case, Jost has given a counterexample to local $\mathcal{L}_+$ connectedness. Consider the space $\mathbb{C}^{12}$ of three four-vectors. Suppose the four-vectors are represented in a basis $e_0, \ldots, e_3$, with components $e_i^\mu = \delta_i^\mu$. Now go over to a new basis

$$f_0 = \frac{1}{\sqrt{2}}(e_0 + e_1), \qquad f_2 = ie_2,$$
$$f_1 = \frac{1}{\sqrt{2}}(e_0 - e_1), \qquad f_3 = ie_3.$$

Let $x = (x^0, x^1, x^2, x^3) \in \mathbb{C}^4$ be a four-vector expressed in the new basis. Then the invariant form becomes

$$2x^0x^1 + (x^2)^2 + (x^3)^2.$$

Consider points of the form

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 1 & 0 & 0 \end{pmatrix}, \qquad K' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & -\beta \\ 1 & 0 & 0 \end{pmatrix},$$

where each column is a four-vector.

Assume that $\mathbb{C}^{12}$ is locally $\mathcal{L}_+$-connected at Z. Then there is a $U(Z, \varepsilon)$ that is $\mathcal{L}_+$-connected in $U'(Z)$. Choose $|\alpha|, |\beta| < \varepsilon/2$. By assumption there is a path $K(t) = \Lambda(t)K$, $0 \leq t \leq 1$, from K to K' in $U'(Z)$. It is easy to show that this path can be brought back into the little group of the vectors (z_1, z_2) by a continuous set of transformations $\Sigma(t) \in \mathcal{L}_+$ near the identity, *i.e.*,

$$\|\Sigma(t) - I\| < B\varepsilon, \quad 0 \leq t \leq 1; \qquad \Sigma(t)\Lambda(t)(z_1, z_2) = (z_1, z_2)$$

where B is a positive, finite number. Thus, letting $\Lambda_1(t) = \Sigma(t)\Lambda(t)$, the path $K_1(t) = \Lambda_1(t)K$ stays within a bounded distance of Z, and

$$\Lambda_1(t) = I/\langle z_1\rangle \oplus \Gamma(t)/\langle z_1\rangle^{\perp}$$

where $\Gamma(t)/\langle z_1\rangle^{\perp}$ is in the little group of the vector $(1,0,0)$ in the three-dimensional space $\langle z_1\rangle^{\perp}$.

One can easily write down this little group explicitly in terms of a single complex parameter ρ [10]:

$$\Gamma(\rho)/\langle z_1\rangle^{\perp} = \begin{pmatrix} 1 & -\frac{1}{2}\rho^2 & -\rho \\ 0 & 1 & 0 \\ 0 & \rho & 1 \end{pmatrix}.$$

Correspondingly, let $K_1(\rho)$ parameterize the orbit of K with respect to this group. Then the distance between $K_1(\rho)$ and K expressed in the standard unitary norm becomes

$$\|K_1(\rho) - K\| = \left(|\rho|^2 \left| \frac{\rho\alpha}{2} + \beta \right|^2 + |\alpha\rho|^2 \right)^{\frac{1}{2}}.$$

We are interested in the minimum over all paths $\rho(t)$, leading from K to K', of the maximum of this distance for each path. By geometrical arguments, one finds that this is $M > |\beta^2/2\alpha|$. By a suitable choice of the ratio β/α, we can make M as big as we like, for any fixed ε, and therefore we get a contradiction with the hypothesis of $\mathcal{L}_+$ connectedness.

(iii) $\underline{\text{Rank } G(Z) = 0}$ We make use of the above mentioned basis $f_0, \ldots, f_3$, and consider again the space $\mathbb{C}^{12}$. The counterexample can be generalized in the following way: Let

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} & 0 \\ 0 & & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\eta \\ i \end{pmatrix} & \beta x \\ 0 & & \beta y \end{pmatrix},$$

$$K' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\eta \\ i \end{pmatrix} & -\beta x \\ 0 & & -\beta y \end{pmatrix};$$

$$x = \frac{-i}{\sqrt{(1+\eta^2)-1}}, \qquad y = \frac{1+\eta}{\sqrt{(1+\eta^2)-1}}.$$

Assume that local $\mathcal{L}_+$ connectedness holds at Z, so that there is a $U(Z,\varepsilon)$ that is $\mathcal{L}_+$-connected in $U'(Z)$. Choose $|\eta|, |\beta|, |\alpha| < \varepsilon/4$.

By assumption there is a path $K(t) = \Lambda(t)K$ from K to K' in $U'(Z)$, and as in (ii) there is another path from K to K',

$$K_1(t) = \Sigma(t)\Lambda(t)K = \Lambda_1(t)K_1$$

that stays within a bounded distance of Z, with the property $\Lambda_1(t)/\langle k_2\rangle = I$. If we choose for $\langle k_2\rangle^\perp$ the new basis $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,x,y)$, then it becomes clear that the problem is reduced to (ii).

4. Dim $Z = 1$

(i) Rank $G(Z) = 1$ The method is the same as in §3.(i). By using Condition I, the problem can be reduced to the case of §2 in $3\mathscr{C}$ dimensions, and $\mathcal{L}_+$ connectedness is proved.

(ii) Rank $G(Z) = 0$ The counterexample can be generalized as follows: Choose a basis so that the invariant form for a four-vector $x = (x^0, x^1, x^2, x^3) \in \mathbb{C}^4$ is $2x^0x^1 + (x^2)^2 + (x^3)^2$. Consider the space $\mathbb{C}^8$ of two four-vectors, and define

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad K' = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & -\beta \\ 0 & 0 \end{pmatrix}, \quad K' = \Lambda K.$$

Assume local $\mathcal{L}_+$ connectedness at Z, and choose $U(Z,\varepsilon)$ to be $\mathcal{L}_+$-connected in $U'(Z)$. Choose $|\alpha|, |\beta| < \varepsilon/2$, and $\alpha = 0$. Then there is a path $K(t) = \Lambda(t)K$ in $U'(Z)$. Because of Condition I, we can take this path to be of the form

$$K(t) = \begin{pmatrix} 1 & \alpha(t) \\ 0 & \alpha \\ 0 & \beta'(t) \\ 0 & \beta''(t) \end{pmatrix} = T[r(t), s(t), p(t)]K,$$

$$T(r,s,p) = \begin{pmatrix} 1 & -rs & -rp & -s/p \\ 0 & 1 & 0 & 0 \\ 0 & s & p & 0 \\ 0 & r & 0 & 1/p \end{pmatrix} \in \mathcal{L}_+(Z_1).$$

Since $\delta(t)$, $\beta'(t)$, $\beta''(t)$ are continuous functions of t, there is a continuous path connecting K and K', in or very near to $U'(Z)$, of the following form

$$K_1(t) = \begin{pmatrix} 1 & \delta(t) \\ 0 & \alpha \\ 0 & \beta(t) \\ 0 & 0 \end{pmatrix}, \qquad \beta(t) = \sqrt{\beta'(t)^2 + \beta''(t)^2}.$$

A little calculation now shows, at least when $\alpha \neq 0$, that this is a continuous path in the little group of the vectors $(1,0,0,0)$ and $(0,0,0,1)$, connecting K and K', so that we are reduced to the counterexample of §3.(ii). □(RS)

V. Conclusion

By applying certain of Stapp's other results, including a weak, but nontrivial, local property of $\mathcal{L}_+$ orbits, we have shown that Stapp's Theorem is valid for the entire domain of regularity; and by assuming the generalization of that property for the complex group of linear transformations, G(n, C), (Condition I) along with the connectedness of the little groups (Condition II), we have proved local G(n, C) covariance for the analytic continuation of a real-covariant or complex-covariant, tensor-valued function. It follows from the arguments used by Stapp for $\mathcal{L}_+$ that the domain of regularity is a union of G(n, C)-invariant sheets, on each of which the function is G(n, C)-covariant.

We have also seen how Jost's counterexample clarifies the local properties of $\mathcal{L}_+$ orbits.

We want finally to comment on the application of these results to the connected, classical, complex groups of linear transformations, defined by their action on $\mathbb{C}^n$: GL(n, C), the general linear groups; SL(n, C), the special linear groups (unimodular matrices); $O_+(n, C)$, the proper, orthogonal groups; and Sp(n, C), the symplectic groups.

All of these groups have an analytic parameterization of a neighborhood of the identity [8], so that Lemma A, or weak G(n, C) covariance, applies. Conditions I and II are easy to show for GL(n, C) and SL(n, C), and we have verified them as well for $O_+(n, C)$, so that Stapp's theorem, in the form just stated, holds for these cases. We expect the conditions to be valid for Sp(n, C) as well, and Stapp's Theorem to hold for any of the classical, connected, complex groups of linear transformations of $\mathbb{C}^n$.

These and related questions are being considered in detail by R. Seiler.

As for the local $\mathcal{L}_+$ connectedness property, Stapp's proof suffices as well for $O_+(n, C)$ and points $Z \in \mathbb{C}^{n\ell}$ satisfying

$$\text{Dim } Z = \text{Rank } G(Z), \quad \text{or} \quad n-1.$$

Jost's counterexample certainly generalizes to a large class of cases, but we have not done an exhaustive study.

Appendix: Proof of Lemma A

As a starting point, we take Stapp's Lemma 7:

Lemma. Let f satisfy the conditions of Lemma A. Let $P \in R$ and let $\Lambda(t)$ be any arc in G(n, C), with $0 \le t \le 1$, $\Lambda(0) = I$, such that $\Lambda(t)P \in R$ for all t. Then

$$\Lambda(t) f(P) = f[\Lambda(t)P].$$

We have stated this lemma in a slightly more general form than given by Stapp, because R need not be schlicht, and the group need not be $\mathcal{L}_+$. However, Stapp's proof still suffices. The lemma is almost Lemma A, but we must still show that $\Lambda(t)P \in R$ for *any* choice of $\Lambda(t)$.

First, we give a corollary to the lemma above:

Corollary. Let f satisfy the conditons of Lemma A. Let U and U' be schlicht subdomains of R with $U \subset\subset U'$. Then there is a neighborhood of the identity, $\mathcal{N}(I) \subset \mathrm{G(n, C)}$, such that, if P and $P' \in R$, and $P' = \Lambda P$ with $\Lambda \in \mathcal{N}(I)$, then $\Lambda f(P) = f(\Lambda P)$.

The corollary follows from the lemma by choosing $\mathcal{N}(I)$ to be connected, and small enough so that $\Lambda U \subset U'$ for all $\Lambda \in \mathcal{N}(I)$.

Now to prove Lemma A, let $P_0 \in R$ and let U and U' be schlicht neighborhoods of P_0 with $U \subset\subset U' \subset R$. Let $\mathcal{N}(I) \subset \mathrm{G(n, C)}$ be the neighborhood of the identity corresponding to U in the corollary above. Let $\Lambda(t) \in \mathrm{G(n, C)}$, $0 \leq t \leq 1$, $\Lambda(0) = I$, be any arc. For fixed t define the new function

$$f^t[\Lambda(t)P] = \Lambda(t) f(P).$$

Then f^t is holomorphic in the schlicht domain $U_t = \Lambda(t)U$. We want to show that $f^{t=1}$ is the analytic continuation of f from U to U_t.

Divide the interval $0 \leq t \leq 1$ into small subintervals, by setting

$$t_n = \frac{n}{L}, \qquad n = 0, 1, \ldots, L.$$

Choose the integer L large enough so that

(i) $\Lambda^{-1}(t_{n+1})\Lambda(t_n) \in \mathcal{N}_0(I) \subset \mathcal{N}(I)$, where $\mathcal{N}_0(I)$ is a neighborhood of the identity to be chosen below; this can be done because $\Lambda(t)$ and $\Lambda^{-1}(t)$ are continuous, and hence uniformly continuous on $0 \leq t \leq 1$.

(ii) $U_{n+1} \cap U_n$ is nonempty, where $U_n \equiv U_{t_n}$. This can be done by choosing $\mathcal{N}_0(I)$ small enough.

Now consider $f^n(P)$ and $f^{n+1}(P)$, where $f^n \equiv f^{t_n}$, with $P \in U_n \cap U_{n+1}$. From the definition,

$$f^n(P) = \Lambda_n f(\Lambda_n^{-1} P),$$
$$f^{n+1}(P) = \Lambda_{n+1} f(\Lambda_{n+1}^{-1} P),$$

with $\Lambda_n^{-1} P$ and $\Lambda_{n+1}^{-1} P \in U$. But

$$\Lambda_{n+1}^{-1} P = \Lambda_{n+1}^{-1} \Lambda_n (\Lambda_n^{-1} P).$$

Thus from (i) and the corollary above, it follows that $f^{n+1}(P) = f^n(P)$, and that f^{n+1} is the unique analytic continuation of f^n. This proves Lemma A, part (i), and part (ii) as well, because U_t was chosen to be schlicht.

The above method of proof is of course familiar from the work of Hall and Wightman [3].

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