Notes on the Umezawa-Kållén-Lehmann Representation and the Feynman Propagator*

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*This is a LaTeX version of a 19xx manuscript that was never circulated. It has only cosmetic differences from the original. Missing are literature references and an appendix on the measure property for a symmetric and traceless standard covariant example.

1 Introduction

My aim is to review the properties of the two-point function as I believe them to be known to general field theorists. I do not know of any references where the following remarks have been collected together, and conceivably some of what I have to say about covariant propagators is new.

The wrinkle that I am going to discuss is how to define mathematically the Feynman propagator, which, in the scalar case, is a sum of two products of distributions:

$$\begin{aligned}\Delta_{\mathrm{F}}(x-y) = \Theta(x_0 - y_0)\langle\Omega,\ \phi(x)\phi(y)\,\Omega\rangle \\ + \Theta(y_0 - x_0)\langle\Omega,\ \phi(y)\phi(x)\,\Omega\rangle.\end{aligned} \tag{1.1}$$

I am going to consider two alternative definitions. The first is the regularized T product, developed by K. Hepp. He showed how to define regularized time-ordered, advanced, and retarded products for any number of field operators; and he showed that on-shell scattering aplitudes do not depend on the choice of regularization. The second alternative is the invariant (or covariant, in the case of spin) propagator, which necessarily contains an ambiguity analogous to the choice of regularization in Hepp's method, but which has no effect on the one-particle poles in the propagator. The invariant propagator is due to O. Steinmann, and preceded Hepp's work.

First we recall a few definitions from Schwartz distribution theory; then we discuss the spectral representation of the two-point function for scalar, Hermitian fields and the two ways of defining propagators; and finally we show how to modify the discussion to include spin and nonHermitian fields.

2 Preliminaries on tempered distributions

We are going to see that, although we do not assume it at the outset, we have to deal only with distributions that are tempered. All orders of "integration" are going to be specified in the sense of distributions. So let's briefly review a few concepts.

Tempered distributions are continuous, linear functionals on the Schwartz spaces of test functons $\mathcal{S}(\mathbb{R}^n)$, or simply $\mathcal{S}$, consisting of C^∞, complex functions on $\mathbb{R}^n$ that decrease at infinity, along with all derivatives, faster than any inverse polynomial.[1] A denumerably normal topology is introduced for $\mathcal{S}$, the

[1] We call that *rapid decrease*.

finer points of which do no concern us, but it's good to be aware that there is a notion of convergence whereby we can say that a sequence $\varphi_n \to 0$ in $\mathcal{S}$. A tempered distribution T maps a test function φ into a number, for which one often uses the notations[2]

$$T(\varphi) = \left\langle T,\, \varphi \right\rangle = \int \mathrm{d}x\; T(x)\, \varphi(x). \tag{2.1}$$

Continuity of T is equivalent to the statement

$$|T(\varphi_n)| \to 0 \tag{2.2}$$

whenever $\varphi_n \to 0$ in $\mathcal{S}$.

The essential operations on distributions that we need are differentiation, Fourier transformation, multiplication by a function, and Lorentz transformation. Let's recall how they are defined.

(i) Differentiation: Let $D^m = (\partial/\partial x_1)^{m_1} \dots (\partial/\partial x_n)^{m_n}$. Then $D^m T$ is defined by symbolic integration by parts:

$$\left\langle D^m T,\, \varphi \right\rangle = (-1)^{|m|} \left\langle T,\, D^m \varphi \right\rangle. \tag{2.3}$$

This defines a tempered distribution because $D^m \varphi \in \mathcal{S}$, and the Schwartz topology is such that $\varphi_n \to 0$ implies $D^m \varphi_n \to 0$.

(ii) Fourier transformation: Let

$$(\mathcal{F}\varphi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}k\; \varphi(k)\, e^{-ik\cdot x}. \tag{2.4}$$

We shall use the Minkowski metric in this definition in our applications. Then the inverse Fourier transform is

$$(\mathcal{F}^{-1}\varphi)(x) \equiv \widetilde{\varphi}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}k\; \varphi(k)\, e^{ik\cdot x}. \tag{2.5}$$

[2]Note that although distributions and test functions are complex, a real inner product notation is used here, corresponding to the traditional treatment of linear functionals in the theory of distributions.

It is a familiar fact that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous isomorphisms of $\mathcal{S}$ onto itself. Therefore, the operation $\mathcal{F}T$ defined by

$$\langle \mathcal{F}T,\, \varphi \rangle = \langle T,\, \mathcal{F}\varphi \rangle \tag{2.6}$$

gives a tempered distribution, written symbolically

$$\mathcal{F}T(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}k\; T(k)\, e^{-ik\cdot x}. \tag{2.7}$$

The definition prescribes the "order of integrations" in the expression on the right: integrate first with a test function in x, then do the k "integration".

(iii) Inhomogeneous Lorentz transformations: If $\varphi(x_1, \dots, x_\ell) \in \mathcal{S}(\mathbb{R}^{4\ell})$, we define

$$\varphi^{(a,\Lambda)}(x_1, \dots, x_\ell) = \varphi[\Lambda^{-1}(x_1 - a), \dots, \Lambda^{-1}(x_\ell - a)]\,. \tag{2.8}$$

For any inhomogeneous Lorentz transformation (a, Λ), this is a continuous isomorphism of $\mathcal{S}(\mathbb{R}^{4\ell})$. Then we define $T^{(a,\Lambda)}$ by the formula

$$\langle T^{(a,\Lambda)},\, \varphi \rangle = \langle T,\, \varphi^{(a,\Lambda)^{-1}} \rangle, \tag{2.9}$$

and we write symbolically

$$T^{(a,\Lambda)}(x_1, \dots, x_\ell) = T[\Lambda^{-1}(x_1 - a), \dots, \Lambda^{-1}(x_\ell - a)]\,. \tag{2.10}$$

An invariant distribution is one that satisfies

$$T^{(a,\Lambda)} = T\,. \tag{2.11}$$

(iv) Multiplication by a function: Let $f(x)$ be a C^∞ function that is bounded, with all derivatives, by polynomials. The space of these functions is called $\mathcal{O}_\mathrm{M}(\mathbb{R}^n)$. Define fT by

$$\langle fT,\, \varphi \rangle = \langle T,\, f\varphi \rangle. \tag{2.12}$$

It is worth pointing out that for any fixed tempered distribution, this definition can be extended to a much larger class of functions. An extreme example occcurs when T is a test function. Then f can be any locally integrable, almost everywhere polynomially bounded function. More generally, any tempered distribution has a representation as a finite sum

$$T = \sum_n \mathcal{P}_n h_n \,, \tag{2.13}$$

where $\mathcal{P}_n$ is a polynomial in derivatives, and h_n is a locally integrable, polynomial bounded function.[3] Thus, it is not hard to see that fT defines a tempered distribution whenever $D^m f$ is locally integrable and polynomial bounded for all derivatives that appear on f in the integration by parts that moves the derivatives off of the functions h_n.

We are especially interested in the case where f is a Heaviside function. Suppose $T(x)$ is a distribution of one variable, and let $\Theta(x)$ be the locally integrable, almost everywhere defined and bounded function

$$\Theta(x) = \begin{cases} 1\,, & x > 0\,, \\ 0\,, & x < 0\,. \end{cases} \tag{2.14}$$

Of course $\Theta(x)$ defines a tempered distribution, too.

We call $\Theta(x)T(x)$ the *primitive product* of Θ and T when it is defined in the sense described above, and we want to know some conditions on T so that the primitive product is defined. From what we said above, the condition is that T be a locally integrable function at the origin.

An equivalent criterion for the primitive product to exist is that the weak limit of regularized products should exist. Let's explain what that means. First, we say that the weak limit of a sequence of tempered distributions exists when $\lim \langle T_n, \varphi \rangle$ exists for all test functions. It is a famous theorem that the weak limit defines a tempered distribution whenever it exists.[4] Next, any distribution can be regularized by taking its convolution with a function in the Schwartz space $\mathcal{D}$ of C^∞ functions with compact support. The regularized distribution

$$T_\chi(x) = \int \mathrm{d}y\, T(y)\, \chi(x-y) \tag{2.15}$$

is a function from $\mathcal{O}_\mathrm{M}$. It is customary to restrict the regularizing functions χ to be real, nonnegative, and normalized,

$$\int \mathrm{d}x\;\chi(x) = 1\,. \tag{2.16}$$

[3] There is also such a representation with a single, continuous h. The derivative polynomial is then generally of higher order.

[4] A fact that one would like to think has made M. J. Lighthill a lot of money.

In our application, we also require χ to be even, $\chi(x) = \chi(-x)$. If a sequence of χ's converges weakly to the δ function, the corresponding sequence of regularized distributions T_χ converges weakly to T.

We define the regularized product of Θ and T to be $\Theta_\chi T$, which is okay because Θ_χ belongs to $\mathcal{O}_\mathrm{M}$. The following theorem is probably known, so we do not give the proof. In any case, we assert that we can prove it.

Theorem. Let a sequence of Θ_χ's converge weakly to Θ. Then the sequence of regularized products $\Theta_\chi T$ converges weakly if and only if T is locally integrable at the origin. When the limit exists, it is equal to the primitive product:

$$\lim \Theta_\chi T = \Theta T \,. \tag{2.17}$$

I believe, but have not carefully checked, that the analogous theorem holds for the primitive product fT in general.

3 The two-point function

Here we review the spectral representation (UKL representation) for the twofold vacuum expectation value of a Hermitian, spinless field,

$$\Delta_+(x-y) = \langle \Omega,\, \phi(x)\phi(y)\,\Omega\rangle. \tag{3.1}$$

We assume the Wightman axioms, except we do not need local commutativity. We do not demand that the field be tempered. The crucial assumptions are:

(i) Lorentz invariance (proper, orthochronous transformations).

(ii) Translation invariance.

(iii) Existence of an invariant vacuum.

(iv) Energy-momentum spectrum in the forward light cone, $\overline{\mathrm{V}}_+$.

(v) The field is an operator-valued distribution.

(vi) Positive, Hilbert space metric.

At first, we know only that Δ_+ is a Lorentz invariant distribution depending only on $x-y$. However, positivity of the metric in Hilbert space and Hermiticity of ϕ imply that

$$\iint \mathrm{d}x\, \mathrm{d}y\; \Delta_+(x-y)\, \overline{\varphi(x)}\, \varphi(y) \geq 0 \tag{3.2}$$

for any $\varphi \in \mathcal{D}$. That is just the definition of a *distribution of positive type*, and a theorem of L. Schwartz (generalization of Bochner's theorem) says that therefore:

(i) Δ_+ (x) is a *tempered* distribution.

(ii) The Fourier transform of $\Delta_+(x)$ is a positive, tempered measure (continuous linear functional on continuous functions of rapid decrease).

By standard field theory techniques, the measure

$$\rho(p) = \frac{1}{(2\pi)^2} \int \mathrm{d}x\; e^{ip\cdot x} \Delta_+(x) \tag{3.3}$$

can be written in terms of the spectral resolution of the unitary translation operators

$$U(a) = e^{iP\cdot a} = \int \mathrm{d}E(p)\; e^{ip\cdot a}, \tag{3.4}$$

$$\rho(p) = (2\pi)^2 \langle \Omega,\; \phi(0) \frac{\mathrm{d}E(p)}{\mathrm{d}p}\, \phi(0)\, \Omega \rangle. \tag{3.5}$$

Indeed, we could have used this formula to show that $\rho(p)$ is a measure in the first place, and we are going to use that fact in the next chapter.

This expression and the spectrum condition make it clear that the support of ρ is in the forward light cone; and since ρ is also Lorentz invariant, we can write

$$\rho(p) = \Theta(p_0)\, \widehat{\rho}(p^2)\,, \tag{3.6}$$

giving us finally the UKL representation

$$\begin{aligned} \Delta_+(x) &= \frac{1}{(2\pi)^2} \int \mathrm{d}m^2 \frac{\mathrm{d}^3 p}{2\omega_m}\, \Theta(m^2)\, \widehat{\rho}(m^2)\, e^{-i\widehat{p}\cdot x}, \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2)\, \Delta_+(x, m^2)\,, \end{aligned} \tag{3.7}$$

where $\omega_m = \sqrt{m^2+p^2}$, $\widehat{p} = (\omega, \mathbf{p})$, and $\Delta_+(x, m^2)$ is the two-point function for a free field of mass m.

Again, this formula is to be understood as meaning: integrate first with a test function $\varphi(x)$ in $\mathcal{S}$, then integrate the result over $\mathrm{d}^3 p$, then over $\mathrm{d}m^2$:

$$\int \mathrm{d}x \; \Delta_+(x)\varphi(x) = \int \mathrm{d}m^2 \int \frac{\mathrm{d}^3 p}{2\omega_m} \, \Theta(m^2)\, \widehat{\rho}(m^2)\, \widetilde{\varphi}(-\widehat{p})\,. \tag{3.8}$$

Temperedness of the measure means that for some integer N, the following integral is well defined:

$$\int_0^\infty \mathrm{d}m^2 \, \frac{\widehat{\rho}(m^2)}{(1+m^2)^N} < \infty\,. \tag{3.9}$$

We emphasize that the UKL representation is *always* well defined as a tempered distribution in x space, and as a tempered distribution in p space, no matter how large the inverse power $(1+m^2)^{-N}$ needed to make the integral over the spectral measure converge.

4 The Feynman propagator

We want to give a meaning to the Feynman propagator

$$\Delta_{\mathrm{F}}(x) = \Theta(x_0)\, \Delta_+(x) + \Theta(-x_0)\, \Delta_+(-x)\,. \tag{4.1}$$

In general, the primitive product $\Theta(x_0)\Delta_+(x)$ does not exist; but under certain conditions on the spectral measure ρ, it does. When the primitive product does not exist, we are going to provide two alternatives.

4.a Regularized propagator

The first, due to K. Hepp [], is just to use the regularized propagator

$$\Delta_{\mathrm{F}}^{\chi}(x) = \Theta_\chi(x_0)\, \Delta_+(x) + \Theta_\chi(-x_0)\, \Delta_+(-x)\,. \tag{4.2}$$

In most applications, the result should be independent of regularization. We have the representation

$$\Theta_\chi(x_0)\Delta_+(x) = \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}p \; e^{-ip\cdot x} \, \frac{\sqrt{2\pi}\, \widetilde{\chi}(p_0-\omega_m)}{2\omega_m(p_0 - \omega_m + i\varepsilon)}\,, \tag{4.3}$$

$$\Delta_{\mathrm{F}}^{\chi}(x) = \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2) \int \mathrm{d}p\; e^{-ip\cdot x}\, \frac{\sqrt{2\pi}}{2\omega_m} \times \left[\frac{\widetilde{\chi}(p_0-\omega_m)(p_0+\omega_m) + \widetilde{\chi}(-p_0-\omega_m)(\omega_m-p_0)}{p^2 - m^2 + i\varepsilon} \right] \tag{4.4}$$

Because χ is normalized to give $\sqrt{2\pi}\,\widetilde{\chi}(0) = 1$, the residues of any one-particle poles in the propagator are independent of χ.

In p space the order of integration in the above formulas is prescribed as follows: smear first in p_0, then in $\mathbf{p}$, and integrate the result over the measure $\widehat{\rho}$. It is not too hard to verify that the effect of the regularization $\widetilde{\chi}$ is to give a rapidly decreasing function of m^2, after the p_0 and $\mathbf{p}$ integrations. It is intuitively reasonable, and can be shown, that the condition for the limit as $\sqrt{2\pi}\,\widetilde{\chi} \to 1$ to exist, and hence for the primitive propagator to exist, is that the result of smearing the distribution

$$\frac{1}{2\omega_m} \cdot \frac{1}{p_0 - \omega_m + i\varepsilon}$$

with a test function in p_0 and $\mathbf{p}$ be integrable with the measure $\widehat{\rho}$. We show in the Appendix that the condition one gets by counting powers of m^2 is correct, and that we have the result:

Theorem. The primitive propagator defined by

$$\Delta_{\mathrm{F}}(x) = \lim_{\chi\to\delta} \Delta_{\mathrm{F}}^{\chi}(x) \tag{4.5}$$

exists if and only if

$$\int_0^\infty \mathrm{d}m^2\, \frac{\widehat{\rho}(m^2)}{1 + m^2} < \infty\,. \tag{4.6}$$

In that case we have the invariant spectral representation

$$\begin{aligned} \Delta_{\mathrm{F}}(x) &= \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2) \int \mathrm{d}p\; e^{-ip\cdot x}\, \frac{1}{p^2 - m^2 + i\varepsilon} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2)\, \Delta_{\mathrm{F}}(x, m^2)\,. \end{aligned} \tag{4.7}$$

4.b Sharp propagator

In case the primitive propagator does not exist, an alternative to Hepp's regularization is to answer the general question: if the product $\Theta(x_0)\,\Delta_+(x)$ could be sharply defined, what minimum properties ought the sharp propagator to have? It seems natural to demand the following:

(i) Δ_{F} should be an invariant, tempered distribution;

$$\text{(ii) } \Delta_{\mathrm{F}}(x) = \begin{cases} \Delta_+(x) & \text{for } x_0 > 0, \\ \Delta_+(-x) & \text{for } x_0 < 0. \end{cases} \tag{4.8}$$

We do not further specify what happens at the singular point, $x_0 = 0$, where the primitive propagator may be undefined.

We are going to see that this problem has solutions, and we are going to find them all. The problem was first posed and solved in this way by O. Steinmann.

First, let's assume a solution exists, and ask how unique it is. Let Δ_{F} and Δ'_{F} be two solutions. Then

$$\xi(x) \equiv \Delta_{\mathrm{F}}(x) - \Delta'_{\mathrm{F}}(x) \tag{4.9}$$

is an invariant distribution, with support contained in the spacelike hyperplane $x_0 = 0$. Since $\xi(x)$ is invariant, its support is invariant, too; and the only invariant subset of $x_0 = 0$ is the point $x = 0$. There is a theorem of L. Schwartz which says that a distribution with support only at a point is a finite sum of derivatives of δ functions. It can be shown that an invariant sum of derivatives of δ functions is a sum of invariant derivatives of δ functions, so

$$\xi(x) = \mathcal{P}(\square)\,\delta(x)\,, \tag{4.10}$$

where $\mathcal{P}$ is some polynomial.

Clearly, then, the sharp propagator can be defined in p-space only up to polynomials in p^2.

Now let's find the spectral representation of all such propagators (and incidentally prove their existence). We have already seen that the primitive product $\Theta\,\Delta_+$ is defined if and only if $\int \mathrm{d}m^2\,\widehat{\rho}(m^2)/(1+m^2) < \infty$. Given the two-point function Δ_+, and its spectral measure $\widehat{\rho}$, we can define a new spectral measure

$$\widehat{\rho}_M \equiv \frac{\widehat{\rho}(m^2)}{(1+m^2)^M}\,. \tag{4.11}$$

More generally, we can define

$$\widehat{\rho}_Q \equiv \frac{\widehat{\rho}(m^2)}{Q(m^2)}\,, \tag{4.12}$$

where Q is any polynomial having no zeros in the support of ρ. Now if we put

$$\Delta_+^Q(x) = \frac{1}{(2\pi)^2}\int \mathrm{d}m^2 \frac{\mathrm{d}^3p}{2\omega_m}\,\frac{\Theta(m^2)\,\widehat{\rho}(m^2)}{Q(m^2)}\,e^{-i\widehat{p}\cdot x}, \tag{4.13}$$

we get the distribution identity

$$Q(-\Box)\,\Delta_+^Q(x) = \Delta_+(x)\,. \tag{4.14}$$

It is always possible to choose the degree of Q large enough so that

$$\int \mathrm{d}m^2 \frac{\widehat{\rho}(m^2)}{(1+m^2)Q(m^2)} < \infty\,, \tag{4.15}$$

because $\widehat{\rho}$ is tempered. For any such Q, the primitive product $\Theta\,\Delta_+^Q$ exists, and we may define a sharp, invariant propagator by writing

$$\begin{aligned}\Delta_{\mathrm{F}}(x) &= Q(-\Box)\,\left[\Theta(x_0)\,\Delta_+^Q(x) + \Theta(-x_0)\,\Delta_+^Q(-x)\right]\\ &= \frac{i}{(2\pi)^3}\int_0^\infty \mathrm{d}m^2 \int \mathrm{d}p\; e^{-ip\cdot x}\,\widehat{\rho}(m^2)\,\frac{Q(p^2)}{Q(m^2)}\,\frac{1}{p^2-m^2+i\varepsilon}\\ &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}m^2\,\frac{\widehat{\rho}(m^2)}{Q(m^2)}\,Q(-\Box)\,\Delta_{\mathrm{F}}(x,m^2)\,.\end{aligned} \tag{4.16}$$

It is straightforward to check that any two such representations, with Q's of sufficiently high degree, indeed differ by a polynomial in p^2. It is less trivial, but still true, that any polynomial $R(p^2)$ can be written

$$R(p^2) = \frac{i}{(2\pi)^3}\int \mathrm{d}m^2\,\widehat{\rho}(m^2)\left[\frac{Q(p^2)}{Q(m^2)} - \frac{Q'(p^2)}{Q'(m^2)}\right]\frac{1}{p^2-m^2}\,, \tag{4.17}$$

where Q and Q' may be chosen to have any degree greater than a fixed N large enough to make the integrals converge, and where both Q and Q' are bounded away from zero for $p^2 \geq 0$. We show that in an appendix.

In other words, the representation (4.16) gives *all* solutions for the invariant propagator.

Note that, just as for Hepp's regularization, any of these propagators has the same poles and residues at any elementary particle masses appearing as δ functions in $\widehat{\rho}$.

Note also that, if we put no further restriction on what we want the sharp propagator to mean, there is no avoiding the polynomial ambiguity; it is present even if the primitive propagator exists. Naturally, one favors representations with the simplest polynomials Q of smallest degree, which picks out the primitive propagator, when it exists.

For the free field of mass $\mu \geq 0$, corresponding to the measure

$$\widehat{\rho} = (2\pi)^2\, \delta(m^2 - \mu^2)\,, \tag{4.18}$$

the primitive propagator certainly exists, because $\widehat{\rho}$ has compact support. Nevertheless, the general propagator for the spinless free field contains the polynomial ambiguity:

$$\widetilde{\Delta}_{\mathrm{F}}(p, \mu^2) = \frac{i}{2\pi}\, \frac{Q(p^2)}{Q(\mu^2)}\, \frac{1}{p^2 - m^2 + i\varepsilon}\,, \tag{4.19}$$

where Q is any polynomial as long as $Q(\mu^2) \neq 0$.

5 Covariant propagators

If we let the field operators transform according to some nontrivial, finite–dimensional representation of SL(2, C), that is, give them a spin content, and if we allow them to be non-Hermitean, the treatment we gave above still goes through with straightforward modifications; but there are two technical points that need some attention.

First, if we allow non-Hermitean fields, or if we want to discuss the two-point function for two different fields, we no longer have distributions of positive type in x-space; and we need a further argument to be sure that we have tempered measures in p-space.

After that is settled, we have to deal with covariant, rather than invariant, measures; and it is customary to decompose the covariant measure into a finite sum over "kinematical-singularity free", standard covariant polynomials. We then want to be sure that the coefficients of these polynomials are invariant *measures* in m^2, and not something more singular. This problem has been solved for covariant, analytic functions of one four-vector by K. Hepp and H. Araki, and there is some lore on the problem for distributions, but I am not aware that it is written down.[5] At any rate, the present problem is such a small strain on the present state of the art that we may consider it an exercise, which I shall work out below, with the understanding that I am probably not the first to do so.

5.a Reduction to tempered measures

Before worrying about covariance, let's check that we have tempered measures in p-space. Suppose we have translation invariant fields $\psi^*(x)$ and $\phi(x)$, which may be particular components of some spinors or vectors, and which are not necessarily Hermitean, tempered, or spacelike commutative or anticommutative. We already mentioned in Chapter 3 that we can use the spectral resolution of the translation operator (insertion of a complete set of momentum states) to show that the two-point function is the Fourier transform of a measure:

$$\langle \Omega,\, \psi^*(x)\, \phi(y)\, \Omega\rangle = \int \mathrm{d}p\; e^{-ip\cdot(x-y)}\, \langle \Omega,\, \psi^*(0) \frac{\mathrm{d}E(p)}{\mathrm{d}p}\, \phi(0)\, \Omega\rangle. \tag{5.1}$$

It remains to show that it is tempered.

Consider the smeared fields, e.g.,

$$\phi(f) = \int \mathrm{d}x\; \phi(x)\, f(x)\,, \tag{5.2}$$

for $f \in \mathcal{D}$. From the Schwartz inequality, we get

$$|\langle \Omega,\, \psi(f)^*\, \phi(g)\, \Omega\rangle|^2 \leq |\langle \Omega,\, \psi(f)^*\, \psi(f)\, \Omega\rangle| \cdot |\langle \Omega,\, \phi(g)^*\, \phi(g)\, \Omega\rangle|\,. \tag{5.3}$$

The two-point functions on the r.h.s. are tempered, by the same positivity argument as before, so we learn that

$$\begin{aligned} &\langle \Omega,\, \psi(f)^*\, \phi(g_n)\, \Omega\rangle \longrightarrow 0\,, \\ &\langle \Omega,\, \psi(f_n)^*\, \phi(g)\, \Omega\rangle \longrightarrow 0\,, \end{aligned} \tag{5.4}$$

for any sequences $\{g_n\}$ and $\{f_n\}$ of functions in $\mathcal{D}$ that converge to zero in the topology induced in $\mathcal{D}$ by $\mathcal{S}$.

[5] For example, I know that W. Schneider had some results on covariant distributions of one four-vector early in 1964, which may very well have included the one we need here.

By a standard completion argument ($\mathcal{D}$ is dense in $\mathcal{S}$, and $\mathcal{S}$ is a metric space), we conclude that $\langle\Omega,\ \psi(x)^*\,\phi(y)\,\Omega\rangle$ extends to a tempered distribution separately in x and y, and by the *théorème nucléaire*, to a tempered distribution in x and y together.

We are therefore assured that

$$\rho(p) = \frac{1}{(2\pi)^2} \int \mathrm{d}p\; e^{ip\cdot x}\,\langle\Omega,\ \psi^*(x)\,\phi(0)\,\Omega\rangle \tag{5.5}$$

is a tempered measure, generally complex if $\psi \neq \phi$. The spectrum condition still implies that its support is for $p \in \overline{\mathrm{V}}_+$.

5.b Irreducible spinor fields without discrete zero mass

We restrict ourselves for now to covariant fields with a spinor index corresponding to irreducible representations of the type $(S,0)$, labeled, e.g., $\phi(x)_\alpha$, $\alpha = S, S{-}1, \ldots -S$, or of the type $(0,S)$ labeled $\phi(x)^{\dot\alpha}$. The transformation laws are

$$\begin{aligned} U(A)\,\phi(x)_\alpha\,U(A)^{-1} &= \mathrm{D}^S(A)_\alpha{}^\beta\,\phi(\Lambda x)_\beta\,, \\ U(A)\,\phi(x)^{\dot\alpha}\,U(A)^{-1} &= \mathrm{D}^S(A^*)^{\dot\alpha}{}_{\dot\beta}\,\phi(\Lambda x)^{\dot\beta}, \end{aligned} \tag{5.6}$$

where $A \in \mathrm{SL}(2,\mathrm{C})$, $\Lambda = \Lambda(A) \in \mathrm{L}^\uparrow_+$, $U(A)$ is the unitary representation of $\mathrm{SL}(2,\mathrm{C})$ on the Hilbert space appearing in the Wightman axioms, and $\mathrm{D}^S(A)$ is a $(2S{+}1)$-dimensional representation of $\mathrm{SL}(2,\mathrm{C})$. Spinor indices are raised and lowered by operating with the symbols $\mathrm{D}^S(\varepsilon)$,

$$\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.7}$$

For the massive free field, ϕ_α or $\phi^{\dot\alpha}$ could be half the components of a Streater-Wightman-Ruelle-Joos-(Stora-Williams, unpublished)-Weinberg, $2(2S{+}1)$-component field.

The changes that have to be made when the fields carry any finite-dimensional representation of $\mathrm{SL}(2,\mathrm{C})$, satisfying perhaps some subsidiary condition, are not serious for the two-point function, and we are going to explain them later.

For now, the spectral measures are basically of the types,

$$\rho = \Theta(p_0)\,\rho(p)_{\alpha\dot\beta}\,, \tag{5.8a}$$

corresponding to $\langle \Omega,\ \psi^*(x)_\alpha\, \phi(y)_{\dot\beta}\, \Omega\rangle$, and

$$\rho = \Theta(p_0)\, \rho(p)_{\alpha\beta}\,, \tag{5.8b}$$

corresponding to $\langle \Omega,\ \psi^*(x)_\alpha\, \phi(y)_\beta\, \Omega\rangle$, plus the two complex conjugate types. The measure ρ is mainifestly covariant, and as in the invariant case, it is convenient to write it in terms of the variables $(m^2, \widehat{p})$:

$$\mathrm{d}p\, \Theta(p_0)\, \rho(p) = \mathrm{d}m^2\, \frac{\mathrm{d}^3 p}{2\omega_m}\, \Theta(m^2)\, \rho(m^2, \widehat{p})\,. \tag{5.9}$$

Then $\Theta(m^2)\, \rho(m^2, \widehat{p})$ is still a tempered, covariant measure in the new variables, because the transformation $p \leftrightarrow (m^2, \widehat{p})$ preserves continuous functions of rapid decrease with support in $\overline{\mathrm{V}}_+$.[6]

Before going further, we want to separate out from ρ any discrete piece it might have at $m^2 = 0$; i.e., we remove any $\delta(m^2)$ singularity. That part corresponds to the two-point function for free, massless fields; and it deserves to be treated separately because of the interplay between manifest covariance and gauge invariance. We do allow the spectrum to go all the way down to $m^2 = 0$, but from now on we assume that the light cone has ρ-measure zero.

The following dicussion shows in passing the basic fact that ρ vanishes unless the fields ϕ and ψ have the same spins, $S = S'$.

Fubini's theorem on iterated integration permits us to consider $\rho(m^2, \widehat{p})$ as a covariant measure in $\widehat{p}$, on the mass shell, for any fixed m^2 in the support of ρ. We may neglect the point $m^2 = 0$, because it now has measure zero. We could apply the Jost-Hepp theorem to conclude that ρ is a covariant, C^∞ function of $\mathbf{p}$ for almost every fixed m^2, but we can also see this directly. It is well known that every SL(2, C)-covariant function of $\widehat{p}$ with nonzero mass and the transformation law of ρ is proportional to

$$\mathrm{D}^S(\varepsilon)\,, \qquad \text{for} \quad \rho_{\alpha\beta}\,, \tag{5.10a}$$

$$\mathrm{D}^S(\widehat{p}\cdot\sigma)\,, \quad \text{for} \quad \rho_{\alpha\dot\beta}\,, \tag{5.10b}$$

where $\widehat{p}\cdot\sigma = \omega_m I + \widehat{\mathbf{p}}\cdot\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}$ is the Pauli three-vector. We can prove the same thing for measures (or even distributions), on the mass shell.

For example, for $\rho_{\alpha\dot\beta}$ we get from covariance:

$$\rho(\widehat{p}) = \mathrm{D}^S\Big(\sqrt{p\cdot\sigma/m}\Big)\, \rho(m, 0, 0, 0)\, \mathrm{D}^S\Big(\sqrt{p\cdot\sigma/m}\Big)\,, \tag{5.11}$$

[6]One of Gårding's isomorphisms.

where $\mathrm{D}^S\left(\sqrt{p\cdot\sigma/m}\right)$ is the SL(2, C) representation of the boost from $(m, 0, 0, 0)$ to $\widehat{p}$. This expression is well-defined because $\mathrm{D}^S\left(\sqrt{p\cdot\sigma/m}\right)$ is in $\mathcal{O}_{\mathrm{M}}(\mathbf{p})$ for fixed $m \neq 0$. Rotation invariance then implies that $\rho(m, 0, 0, 0)$ is proportional to $\delta_{\alpha\dot{\beta}}$, and we can write

$$\widehat{\rho}(m^2, \widehat{p})_{\alpha\dot{\beta}} = \widehat{\rho}(m^2)\,\mathrm{D}^S(\widehat{p}\cdot\sigma)_{\alpha\dot{\beta}}\,. \tag{5.12}$$

A similar argument gives

$$\widehat{\rho}(m^2, \widehat{p})_{\alpha\beta} = \widehat{\rho}(m^2)\,\mathrm{D}^S(\varepsilon)_{\alpha\beta}\,. \tag{5.13}$$

We have been careful in these expressions to make the coefficient of $\widehat{\rho}$ a polynomial in $\widehat{p}$, because we wanted to arrange that $\widehat{\rho}$ be a measure in m^2 even near $m^2 = 0$. To check that $\widehat{\rho}$ is a tempered measure, it is sufficient to find continuous, rapidly decreasing functions $f(\widehat{p})^{\dot{\beta}\alpha}$ and $g(\widehat{p})^{\beta\alpha}$ such that the integrals

$$\begin{aligned}\widehat{f}(m^2) &\equiv \int \frac{\mathrm{d}^3 p}{2\omega_m}\ \mathrm{Tr}\left[\mathrm{D}^S(\widehat{p}\cdot\sigma)\, f(\widehat{p})\right]\\ \widehat{g}(m^2) &\equiv \int \frac{\mathrm{d}^3 p}{2\omega_m}\ \mathrm{Tr}\left[\mathrm{D}^S(\varepsilon)\, g(\widehat{p})\right]\end{aligned} \tag{5.14}$$

are continuous functions of m^2, uniformly bounded away from zero; for then

$$\chi(m^2, \widehat{p})^{\dot{\beta}\alpha} = h(m^2)\,\frac{f(\widehat{p})^{\dot{\beta}\alpha}}{\widehat{f}(m^2)}$$

and

$$\chi(m^2, \widehat{p})^{\beta\alpha} = h(m^2)\,\frac{g(\widehat{p})^{\beta\alpha}}{\widehat{g}(m^2)}$$

are continuous and rapidly decreasing whenever $h(m^2)$ is continuous and rapidly decreasing; and

$$\int_0^\infty \mathrm{d}m^2\,\widehat{\rho}(m^2)\,h(m^2) = \int \mathrm{d}m^2 \frac{\mathrm{d}^3 p}{2\omega_m}\ \mathrm{Tr}(\rho\chi) \tag{5.15}$$

defines $\widehat{\rho}$ as a tempered measure.

Examples of functions f and g that do the job are

$$
\begin{aligned}
g^{\beta\alpha} &= 2\omega_m\, G(\mathbf{p})\, \mathrm{D}^S(\varepsilon^{-1})^{\beta\alpha}, \\
f^{\dot\beta\alpha} &= 2\omega_m\, F(\mathbf{p})\, \delta^{\dot\beta\alpha},
\end{aligned}
\tag{5.16}
$$

where G and F are any continuous, nonzero, rapidly decreasing, positive functions. Then we get

$$
\begin{aligned}
\int \frac{\mathrm{d}^3p}{2\omega_m}\ \mathrm{Tr}\left[\mathrm{D}^S(\varepsilon)\, g\right] &= (2S+1)\int \mathrm{d}^3p\ G(\mathbf{p}) \\
&= \text{const} > 0\,;
\end{aligned}
\tag{5.17a}
$$

$$
\begin{aligned}
\int \frac{\mathrm{d}^3p}{2\omega_m}\ \mathrm{Tr}\left[\mathrm{D}^S(\widehat{p}\cdot\sigma)\, f\right] &= (2S+1)\int \mathrm{d}^3p\ (\omega_m)^{2S}\ F(\mathbf{p}) \\
&\geq (2S+1)\int \mathrm{d}^3p\ |\mathbf{p}|^{2S}\ F(\mathbf{p}) \\
&= \text{const} > 0\,.
\end{aligned}
\tag{5.17b}
$$

Therefore, we have proved the UKL representation for the two-point function:

$$
S_+(x-y) = \langle \Omega,\, \psi^*(x)\, \phi(y)\, \Omega \rangle\,, \tag{5.18a}
$$

$$
\begin{aligned}
S_+(x) &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2) \int \frac{\mathrm{d}^3p}{2\omega_m}\ e^{-i\widehat{p}\cdot x}\, \mathcal{P}(\widehat{p}) \\
&= \frac{\mathcal{P}(i\partial)}{(2\pi)^2}\int_0^\infty \mathrm{d}m^2\, \widehat{\rho}(m^2)\, \Delta_+(x, m^2)\,,
\end{aligned}
\tag{5.18b}
$$

where $\widehat{\rho}$ is a tempered measure (with no $\delta(m^2)$ piece, by hypothesis), and $\mathcal{P}$ is the appropriate standard covariant polynomial

$$
\mathcal{P}(\widehat{p}) = \begin{cases} \mathrm{D}^S(\widehat{p}\cdot\sigma) & \text{for} \quad S_{+\alpha\dot\beta}\,, \\ \mathrm{D}^S(\varepsilon) & \text{for} \quad S_{+\alpha\beta}\,. \end{cases} \tag{5.19}
$$

We argue later that this representation is valid for fields that carry any representation of SL(2, C) having irreducible spin content (for example, a symmetric, traceless, divergenceless tensor), at least if the subsidiary conditions are local, by just putting the correct standard covariant for $\mathcal{P}$.

5.c Irreducible propagators

Before we can discuss the T product, we have to know how $\langle\Omega,\, \psi^*(x)\,\phi(y)\,\Omega\rangle$ is related to $\langle\Omega,\, \phi(y)\,\psi^*(x)\,\Omega\rangle$. We proceed by invoking causal commutation or anticommutation rules for the first time, and writing a UKL representation for each of the two-point functions. It is then "straightforward" to show that their measures have to be the same, up to a sign, and we get

$$\langle\Omega,\, \psi^*(x)_\beta\, \phi(y)_{\dot\alpha}\, \Omega\rangle = (-1)^{2S}\, S_+(y-x)_{\alpha\beta}\,, \tag{5.20a}$$

$$\langle\Omega,\, \phi(y)_{\dot\beta}\, \psi^*(x)_\alpha\, \Omega\rangle = S_+(y-x)_{\alpha\dot\beta}\,. \tag{5.20b}$$

Having said that, we are ready to adapt the theory of invariant propagators to the covariant case. The first remark is that the condition for the existence of the primitive propagator can obviously get modified when $\mathrm{D}^S(\widehat{p}\cdot\sigma)$ is present. But even then, the propagator for fermions is better behaved than the individual terms $\Theta\, S_+$ in its definition:

$$\begin{aligned} \widehat{S}_\mathrm{F}(x) &\equiv \Theta(x_0)\, S_+(x) + \sigma\Theta(-x_0) S_+(-x)\,, \\ \sigma &= \begin{cases} 1 & \text{for} \quad \widehat{S}_{\mathrm{F}\alpha\beta}\,, \\ (-1)^{2S} & \text{for} \quad \widehat{S}_{\mathrm{F}\alpha\dot\beta}\,. \end{cases} \end{aligned} \tag{5.21}$$

To see that in general, we supply an inverse polynomial $Q(m^2)^{-1}$ in the measure $\widehat{\rho}$ to get a well-defined primitive propagator, and a derivative $Q(-\Box)$, to get finally the representation

$$\begin{aligned} \widehat{S}_\mathrm{F} = \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}p\, \frac{Q(p^2)}{Q(m^2)}\, \frac{\widehat{\rho}(m^2)}{p^2 - m^2 + i\varepsilon}\, e^{-ip\cdot x} \\ \times \frac{1}{2\omega_m} \Big[(p^0 + \omega_m)\, \mathcal{P}(\widehat{p}) - (p^0 - \omega_m)\, \mathcal{P}(-\widetilde{\widehat{p}}) \Big]\,, \end{aligned} \tag{5.22}$$

where $\widetilde{\widehat{p}} \equiv (\omega, -\mathbf{p})$.

If $\mathcal{P} = \mathrm{D}^S(\varepsilon)$, the condition for the existence of this representation is the same as before. If $\mathcal{P} = \mathrm{D}^S(\widehat{p}\cdot\sigma)$, the condition for existence for bosons is the same for $\widehat{S}_\mathrm{F}$ and $\Theta\, S_+$:

$$\int_0^\infty \mathrm{d}m^2\, \frac{\widehat{\rho}(m^2)\, m^{2S}}{Q(m^2)\,(1+m^2)} < \infty\,; \quad \text{(bosons)} \tag{5.23}$$

while for fermions, the existence condition for $\Theta\, S_+$ is the same as that above; but for $\widehat{S}_\mathrm{F}$ it is

$$\int_0^\infty \mathrm{d}m^2\, \frac{\widehat{\rho}(m^2)\, m^{2S-1}}{Q(m^2)\,(1+m^2)} < \infty\,. \quad \text{(fermions)} \tag{5.24}$$

These estimates can be seen intuitively by counting the leading powers of ω_m in Eq. (5.22), and they can be justified rigorously.

It is important to note that $\mathcal{P}(\widehat{p})$ is restricted to the mass shell. If $\mathcal{P}(\widehat{p})$ has a nontrivial p dependence, as in the case $\mathcal{P} = \mathrm{D}^S(\widehat{p}\cdot\sigma)$, $S > \frac{1}{2}$, we have the well-known (Schwinger) phenomenon that $\widehat{S}_\mathrm{F}$ is not covariant. It is easy to check that $\widehat{S}_\mathrm{F}$ is covariant when $S = \frac{1}{2}$.

To get a covariant propagator in all cases, we should have asked the same question as in the invariant case: What are the covariant distributions $S_\mathrm{F}(x)$ that are equal to $S_+(x)$ for $x_0 > 0$ and $S_+(-x)$ for $x_0 < 0$? We certainly get a solution to this problem when we replace $\widehat{S}_\mathrm{F}$ by

$$\begin{aligned} S_\mathrm{F}(x) &= \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2\, \frac{\widehat{\rho}(m^2)}{Q(m^2)} \int \mathrm{d}^4 p\, \frac{Q(p^2)\, \mathcal{P}(p)}{p^2 - m^2 + i\varepsilon}\, e^{-ip\cdot x} \\ &= \mathcal{P}(i\partial)\, \Delta_\mathrm{F}(x)\,, \end{aligned} \tag{5.25}$$

where Δ_F is the invariant propagator corresponding to $\widehat{\rho}$, because then S_F is equal to

$$S_+(x) = \mathcal{P}(i\partial)\, \Delta_+(x) \tag{5.26}$$

when $x_0 > 0$, since $\Delta_\mathrm{F} = \Delta_+$, with a similar statement for $S_+(-x)$ and $x_0 < 0$.

Moreover, we can see that all the ambiguity in S_F is contained in Δ_F, i.e., in the choice of the polynomial Q, by the following argument:

(i) A simple extension of our earlier reasoning shows that the ambiguity in x-space is a covariant derivative of $\delta(x)$ having the same transformation law as Δ_+.

(ii) This yields a covariant polynomial in p-space, which must have the form $R(p^2)\, \mathcal{P}(p)$, where R is a polynomial, by the Hepp-Araki result for covariant polynomials in one four-vector.

(iii) We stated before that any polynomial $R(p^2)$ can be written in the form of Eq. (4.17), which shows that $R(p^2)\,\mathcal{P}(p)$ can be written as the difference of two propagators of the form (5.25).

Let us call the covariant propagator corresponding to $Q \equiv 1$ the *fundamental* propagator, when it exists. We distinguish it from the primitive covariant propagator, obtained by replacing both $\mathcal{P}(\omega, \mathbf{p})$ and $\mathcal{P}(-\omega, \mathbf{p})$ by $\mathcal{P}(p)$ in the primitive (noncovariant) propagator, because the conditions for the existence of the fundamental propagator and the primitive propagator are not the same.

Theorem. The *fundamental* propagator

$$\begin{aligned} S_{\mathrm{F}}(x) &= \frac{i}{(2\pi)^3} \int_0^\infty \mathrm{d}m^2 \ \widehat{\rho}(m^2) \int \mathrm{d}^4 p \ \frac{\mathcal{P}(p)\, e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}m^2 \ \widehat{\rho}(m^2)\, \mathcal{P}(i\partial)\, \Delta_{\mathrm{F}}(x, m^2) \end{aligned} \tag{5.27}$$

exists if and only if

$$\int_0^\infty \mathrm{d}m^2 \, \frac{\widehat{\rho}(m^2)}{1 + m^2} < \infty \, . \tag{5.28}$$

In other words, the condition for the existence of the fundamental propagator is *spin independent*.

The fundamental propagator is equal to the primitive covariant propagator if both exist, i.e., if $\widehat{\rho}$ decreases fast enough, but the fundamental propagator may exist even if the primitive propagator does not.

5.d Free field propagators

Let's restrict these results to massive free fields where $\widehat{\rho} = (2\pi)^2\, \delta(p^2 - \mu^2)$, $\mu \neq 0$. The fundamental and primitive propagators exist for any spin, and the covariant propagator in p-space is

$$\widetilde{S}_{\mathrm{F}}(p, \mu^2) = \frac{i}{2\pi} \cdot \frac{Q(p^2)}{Q(\mu^2)} \cdot \frac{\mathcal{P}(p)}{p^2 - \mu^2 + i\varepsilon} \, , \tag{5.29}$$

where Q is any polynomial with $Q(\mu^2) \neq 0$. The fundamental propagator, with $Q \equiv 1$, has the "best" p-space behavior at infinity.

5.e Contact terms

The noncovariance of $\widehat{S}_{\mathrm{F}}$ has been labeled an "infamous difficulty"[7] in the literature, and it has long been recognized that the thing to do is to add to $\widehat{S}_{\mathrm{F}}$ the noncovariant "contact" terms one gets from the commutator

$$\left[\mathcal{P}(i\partial),\, \Theta(x_0)\right]$$

to get a covariant propagator. There occasionally appears to be some uncertainty about this question, so perhaps it is worth a remark.

The procedure indicated by the commutator above is perfectly unambigous in the language developed so far. The main thing we want to point out is that the operations of multiplying the distributions $(\partial_0)^s\,\Delta_+^Q$ by the distributions $(\partial_0)^r\,\delta(x_0)$ that arise from the Leibniz rule are well defined, if and only if Q is chosen to make $\Theta\,\mathcal{P}(i\partial)\,\Delta_+^Q$ well defined, i.e., if and only if $\widehat{\rho}/Q$ decreases fast enough.

That can be justified rigorously,[8] but we only sketch an argument. The existence of $\Theta\,(\partial_0)^n\Delta_+^Q$ means that $(\partial_0)^n\Delta_+^Q$ is locally integrable at $x_0 = 0$. Hence $(\partial_0)^r\Delta_+^Q$ is locally continuous at $x_0 = 0$ for any $r < n$; and we are free to form the products $[(\partial_0)^r\delta(x_0)]\,\Delta_+^Q$.

In particular, these operations are always allowed when we deal with the primitive propagators for a free field of nonzero mass and any spin.

5.f Generalization to any finite dimensional representation

If the fields carry any finite dimensional representations of SL(2, C), we can immediately do a covariant reduction to irreducible representations by applying the equivalent of Clebsch-Gordan coefficients. The important point is that this reduction involves an invertible transformation by numerical matrices, whose only effect is to give us a multiplicity of fields and two-point functions, each of which can be discussed separately, and all of which obey the axioms we have assumed.

Thus, it is "trivial" to reduce the problem to one where each field carries an irreducible representation of SL(2, C) of type (S_1, S_2), which is a little more general than the types $(S, 0)$ or $(0, S)$ considered so far.

The two-point function S_+ of a pair of such fields has a covariant measure

$$\rho_{\alpha'\dot{\beta}',\alpha\dot{\beta}} = \Theta(p_0)\,\rho(m^2,\widehat{p})_{\alpha'\dot{\beta}',\alpha\dot{\beta}}\,, \tag{5.30}$$

[7] A Schwingerism?

[8] I think. It needs checking.

where all the spins S_1, S_2, S_1', S_2' may be different. It can be reduced further by applying Clebsch-Gordan coefficients separately to the dotted and undotted indices, giving us finally a direct sum of covariant measures $\rho_{\alpha\dot{\beta}}$ carrying irreducible representations of SL(2, C), to which we may apply the theory already developed.

From what we said before, the only nonvanishing irreducible components of ρ are those of type (S, S). This language about the "irreducible components of ρ" can be confusing, because the label S is the spin of the elementary particles described by ρ only in special cases. For example, the measure

$$\widehat{\rho}_{\alpha\beta} = \widehat{\rho}(m^2)\,\mathrm{D}^S(\varepsilon)_{\alpha\beta} \tag{5.31}$$

that we discussed before corresponds to states in the Hilbert space of angular momentum S; but it has a single irreducible component of type $(0, 0)$, in the language above. We are using this language mainly as a device to convince the reader that everything goes through for any finite dimensional representation.

If the original fields are subject to subsidiary conditions involving only covariant polyonomials in derivatives, such as divergence, or Bargmann-Wigner, or Rarita-Schwinger conditions, or if there are symmetry and trace conditions on the indices, that produces relations among the irreducible measures in the two-point function, whose coefficients are polynomials; but it is not a source of any particular trouble.

To carry the discussion over to propagators, all that has to be checked is whether causal commutativity or anticommutativity still produces (5.20b) for the two-point functions of fields in opposite order. That is straightforward to verify;[9] and from then on, everything is the same as before.

In principle, the conditions for existence of the propagators may be different for each irreducible two-point function; but all irreducible pieces connected by a subsidiary condition have to be regularized by a common Q, if any Q is necessary.

5.g Symmetric, traceless, divergenceless tensors

As a practical example, we work out the propagator for two fields that are symmetric, traceless, divergenceless tensors of any rank. This of course includes conserved currents and energy-momentum tensor densities; and it is simple enough, as is any other example with irreducible spin content, that we can treat it directly

[9] But needs checking.

without going through the process just outlined of reducing with Clebsch-Gordan coefficients and imposing the subsidiary conditions.

Namely, excluding any discrete, zero mass in the spectrum, the measure

$$\rho^{(\mu)(\nu)} = \Theta(p_0)\, \rho(m^2, \hat{p})^{(\mu)(\nu)} \tag{5.32}$$

corresponding to the two-point function

$$S_+(x-y)^{(\mu)(\nu)} = \langle \Omega,\, \psi^{*\mu_1\cdots\mu_S}\, \phi^{\nu_1\cdots\nu_S}\, \Omega \rangle \tag{5.33}$$

can be reduced to the earlier case by contracting with the $\mathcal{O}_{\mathrm{M}}(p)$ transformation symbols $\tau^{(\mu)}(\hat{p})_\alpha$, defined up to a sign by:

(i) $\tau^{(\mu)}(p)_\alpha$ is a covariant polynomimal in p;

(ii) $\tau^{(\mu)}(p)_\alpha$ is symmetric and traceless in its tensor indices;

(iii) $p^\mu\, \tau_\mu{}^{\mu_2\cdots\mu_S}(p)_\alpha = 0$;

(iv) $\tau^{(\mu)}(p)_\alpha\, \tau_{(\mu)}(p)^\beta = (p^2)^S\, \delta_\alpha{}^\beta$.

It follows that

$$\frac{G^{(\mu)(\nu)}(p)}{(p^2)^S} \equiv \frac{\tau^{(\mu)}(p)_\alpha\, \tau^{(\nu)}(p)^\alpha}{(p^2)^S}\,. \tag{5.34}$$

is the (Fronsdal) projection operator for the symmetric, traceless, conserved tensors of rank S. The transformation symbol τ can be written down explicitly in terms of Pauli matrices and Clebsch-Gordan coefficients.[10]

Let $\rho_{\alpha\beta} = \tau^{(\mu)}{}_\alpha\, \tau^{(\nu)}{}_\beta\, \rho_{(\mu)(\nu)}$. It is a covariant measure to which we can apply the same discussion as before:

$$\rho_{\alpha\beta} = \hat{\sigma}(m^2)\, \mathrm{D}^S(\varepsilon)_{\alpha\beta}\,, \tag{5.35}$$

where $\hat{\sigma}$ is a tempered measure. Then,

$$\rho(m^2, \hat{p})^{(\mu)(\nu)} = \hat{\rho}(m^2)\, G^{(\mu)(\nu)}(\hat{p})\,, \tag{5.36}$$

where

$$\hat{\rho} = \hat{\sigma}/m^{2S} \tag{5.37}$$

[10]This construction is reviewed in Appendix 2 of [].

is a tempered measure, with the possible exception of a neighborhood of $m^2 = 0$, and where $G(p)$ is the covariant polynomial defined in (5.34) above.

In an appendix, we show that we have chosen the standard covariant G correctly, that indeed $\widehat{\rho}$ is a measure near $m^2 = 0$, too. The argument is quite analogous to that we gave before; and it uses the fact that

$$G^{(0)(0)}(\widehat{p}) = C\,(\mathbf{p}\cdot\mathbf{p})^S, \tag{5.38}$$

where C is a nonzero constant, independent of m^2.

From this point, the discussion precisely parallels our earlier remarks:

(i) Local commutatitivty implies that

$$\langle\Omega,\, \phi^{(\nu)}(y)\,\psi^{*(\mu)}(x)\,\Omega\rangle = S_+(y-x)^{(\mu)(\nu)}. \tag{5.39}$$

(ii) The propagator

$$\widehat{S}_{\mathrm{F}}(x)^{(\mu)(\nu)} = \Theta(p_0)\,S_+(x)^{(\mu)(\nu)} + \Theta(-p_0)\,S_+(-x)^{(\mu)(\nu)} \tag{5.40}$$

has the representation

$$\begin{aligned}\widehat{S}_{\mathrm{F}}(x)^{(\mu)(\nu)} = \frac{i}{(2\pi)^3}\int_0^\infty \mathrm{d}m^2\,\frac{\widehat{\rho}(m^2)}{Q(m^2)}\int \mathrm{d}^4p\,\frac{Q(p^2)}{p^2-m^2+i\varepsilon}\,e^{-ip\cdot x}\\ \times\frac{1}{2\omega_m}\Big[(p^0+\omega_m)\,G^{(\mu)(\nu)}(\omega_m,\mathbf{p})\\ -(p^0-\omega_m)\,G^{(\mu)(\nu)}(-\omega_m,\mathbf{p})\Big]\,.\end{aligned} \tag{5.41}$$

(iii) The covariant propagator has the representation

$$S_{\mathrm{F}}(x)^{(\mu)(\nu)} = \frac{i}{(2\pi)^3}\int_0^\infty \mathrm{d}m^2\,\frac{\widehat{\rho}(m^2)}{Q(m^2)}\int \mathrm{d}^4p\,\frac{Q(p^2\,G^{(\mu)(\nu)})}{p^2-m^2+i\varepsilon}\,e^{-ip\cdot x} \tag{5.42a}$$

$$= \frac{i}{2\pi}\,G^{(\mu)(\nu)}(i\partial)\,\Delta_{\mathrm{F}}(x)\,. \tag{5.42b}$$

We leave it to the reader to read off the conditions on $\widehat{\rho}$ and Q for the representations $\widehat{S}_{\mathrm{F}}$ and S_{F} to exist.

The most important special cases are the conserved vector and second rank tensor, where of course

$$G^{\mu\nu} = p^2\, g^{\mu\nu} - p^\mu\, p^\nu\,, \tag{5.43}$$

$$\begin{aligned} G^{\mu_1\nu_1;\mu_2\nu_2} = {} & \tfrac{1}{2}\, p^4 \left(g^{\mu_1\nu_1}\, g^{\mu_2\nu_2} + g^{\mu_1\nu_2}\, g^{\mu_2\nu_1} - \tfrac{2}{3}\, g^{\mu_1\mu_2}\, g^{\nu_1\nu_2} \right) \\ & - \tfrac{1}{2}\, p^2 \,[p^{\mu_1}\, p^{\nu_1}\, g^{\mu_2\nu_2} + p^{\mu_1}\, p^{\nu_2}\, g^{\mu_2\nu_1} + p^{\mu_2}\, p^{\nu_1}\, g^{\mu_1\nu_2} + p^{\mu_2}\, p^{\nu_2}\, g^{\mu_1\nu_1} \\ & \qquad - \tfrac{2}{3}\,(p^{\mu_1}\, p^{\mu_2}\, g^{\nu_1\nu_2} + p^{\nu_1}\, p^{\nu_2}\, g^{\mu_1\mu_2}) \Big] \\ & + \tfrac{2}{3}\, p^{\mu_1}\, p^{\mu_2}\, p^{\nu_1}\, p^{\nu_2}. \end{aligned} \tag{5.44}$$