Am I Decisive?

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I compute the probability that one's vote is *decisive* in a majority-rule election between two candidates. Here, a decisive vote is defined to be a vote that breaks an exact tie among all the other voters. For simplicity I refer to the set of all the other voters as the *electorate*.

I show two different kinds of estimates of the probability that one's vote is decisive. One is the *purely combinatorial* estimate that one should use when one has no information at all about the likely choices of the electorate. The other is the *conditional* estimate one should use when there is *poll data* that tells one something about the electorate. The two estimates may have very different values. The differences between the estimates crudely measure the effect that information a person has about the distribution of others' vote intentions can have on the person's decision whether to vote. It is important both to keep those effects in mind and to develop ways to study them.

First let's derive the purely combinatorial estimate. To get anywhere, we need to give a precise definition of what an electorate is. All we need to know about an electorate is how many people are in it and how each person in it will vote. Real life is much more complicated, but to focus solely on the problem of estimation we will assume that each and every elector has a firm, unchanging preference for one candidate or the other. Because everyone's vote counts the same, all we really need to know about an electorate is its *vote split*: how many people will vote for one candidate and how many will vote for the other. I use the notation N_j to denote an electorate's vote split, where N is the total number of people and j is the number who will vote for Candidate 1 (the remaining $N - j$ people vote for Candidate 2). One's vote is decisive only in an electorate that has vote split $N_{N/2}$.

The purely combinatorial probability that any particular vote split occurs, given that the electorate has N people, is the ratio of the number of possible electorates of size N that have that vote split to the total number of electorates of size N that are possible. The number of possible electorates that have vote split N_j is given by the binomial coefficient,

$$\binom{N}{j} = \frac{N!}{j!(N-j)!} ,$$

where $N! = N \times N - 1 \times \cdots \times 1$. The total number of possible electorates of size N is the number of patterns in which N people may vote. That number is 2^N (the number 2 raised to the power N). The purely combinatorial probability that vote split N_j occurs is therefore

$$\Pr(N_j) = \binom{N}{j} / 2^N .$$

The purely combinatorial estimate of the probability that one's vote is decisive, in an electorate of size N, is therefore

$$\Pr(N_{N/2}) = 2^{-N} \binom{N}{N/2} .$$

Notice that $\Pr(N_{N/2})$ is the same as the formula derived by Hinich and Munger (1997, 147).

If poll data exist, one should use the data to improve one's estimate of the probability that one is decisive. The question to be answered is, given that the poll data are thus and so, what is the probability that the electorate that was polled is an electorate that has vote split $N_{N/2}$. The poll data come from the actual electorate, that is, from the electorate that actually exists. So the question really is, given that the poll data are thus and so, what is the probability that the actual electorate is one in which one's vote is decisive. One learns from the poll data a bit about the characteristics of the electorate that actually exists.

I use a highly simplified model of how the data in a poll are created. I assume that a poll that uses a sample of n observations is conducted by repeating the following exercise n times: select one person from the electorate, totally at random, with probability $1/N$; record that person's vote intention. The same person may be selected at most one time for inclusion in the poll. Real polls differ from this idealized procedure in many ways. In the idealized model of a poll that I am using, the result is a pair of counts: n_1 records the number of votes in the poll for Candidate 1, and n_2 records the number of votes for Candidate 2, with $n_1 + n_2 = n$. I use the notation B_{n_1,n_2} to denote the poll data.

What we want to compute is a conditional probability, namely, the probability that the actual electorate has vote split $N_{N/2}$ given that the poll result is B_{n_1,n_2}. The notation for that conditional probability is $\Pr(N_{N/2} \mid B_{n_1,n_2})$. We use what's known as Bayes Theorem to compute the conditional probability. Essentially, Bayes Theorem determines a value for the conditional probability by inverting the question that the conditional probability expresses. Bayes Theorem says, consider how likely the poll data are to have occurred if the electorate has vote split $N_{N/2}$, compared to how likely the poll data are to have occurred in all possible electorates. Bayes Theorem uses the conditional probabilities of the *poll data*, given each possible electorate, to compute the conditional probability that the actual electorate has vote split $N_{N/2}$, given the poll data. My simplified model of how the poll data are created implies that the conditional probability of a result B_{n_1,n_2}, given vote split N_j, is

$$\Pr(B_{n_1,n_2} \mid N_j) = \binom{n_1}{n} \left(\frac{j}{N}\right)^{n_1} \left(1 - \frac{j}{N}\right)^{n_2} .$$

Bayes Theorem says

$$\Pr(N_{N/2} \mid B_{n_1,n_2}) = \frac{\Pr(B_{n_1,n_2} \mid N_{N/2}) \Pr(N_{N/2})}{\sum_{j=0}^{N} \Pr(B_{n_1,n_2} \mid N_j) \Pr(N_j)} .$$

Substituting in the formulas for the various probabilities on the righthand side and simplifying gives

$$\Pr(N_{N/2} \mid B_{n_1,n_2}) = \frac{\left(\frac{1}{2}\right)^{n} \binom{N}{N/2}}{\sum_{j=0}^{N} \left(\frac{j}{N}\right)^{n_1} \left(1 - \frac{j}{N}\right)^{n_2} \binom{N}{j}} .$$

Notice that if there are no poll data, the conditional estimate reduces to the purely combinatorial estimate. If there are no poll data, then $n_1 = n_2 = 0$. In that case we have

$$\Pr(N_{N/2} \mid B_{0,0}) = \frac{\binom{N}{N/2}}{\sum_{j=0}^{N} \binom{N}{j}} .$$

But $\sum_{j=0}^{N} \binom{N}{j} = 2^N$.

Table 1 shows the probability that one's vote is decisive, for several electorate sizes and poll results.

Electoral College: The preceding model applies when there is a direct election. If there is an indirect election, such as an American presidential election via the Electoral College, the calculations are different. Consider the Electoral College for a two-candidate election when the unit rule holds: all of a State's electoral votes go to the candidate who wins a majority of the popular vote in the State. For simplicity consider a person's vote to be decisive if two conditions hold: a change in the assignment of the electoral votes of the person's State would swing the outcome (create or break a tie) in the Electoral College; and the person's vote breaks a tie within the State.

Table 1: Probability That One's Vote is Decisive in a Direct Election

		Vote Division in Poll Data (Candidate 1/Candidate 2)			
N	n	.5/.5	.51/.49	.55/.45	.6/.4
10	0	.2461	.2461	.2461	.2461
100	0	.0796	.0796	.0796	.0796
	20	.0873	.0872	.0858	.0816
1,000	0	.0252	.0252	.0252	.0252
	20	.0255	.0255	.0254	.0253
	100	.0265	.0264	.0253	.0221
	200	.0276	.0275	.0234	.0142
	500	.0309	.0299	.0134	.0011
10,000	0	.00798	.00798	.00798	.00798
	20	.00799	.00799	.00799	.00798
	100	.00802	.00802	.00798	.00786
	200	.00806	.00805	.00790	.00745
	500	.00818	.00814	.00726	.00508
	1000	.00837	.00822	.00531	.00136
100,000	0	.002523	.002523	.002523	.002523
	20	.002523	.002523	.002523	.002523
	100	.002524	.002524	.002523	.002519
	200	.002526	.002525	.002521	.002506
	500	.002529	.002528	.002498	.002407
	1000	.002536	.002531	.002413	.002080
1,000,000	0	.00079788	.00079788	.00079788	.00079788
	20	.00079789	.00079789	.00079789	.00079789
	100	.00079792	.00079792	.00079788	.00079776
	200	.00079796	.00079796	.00079780	.00079733
	500	.00079808	.00079804	.00079709	.00079411
	1000	.00079828	.00079812	.00079431	.00078249
10,000,000	0	.00025	.00025	.00025	.00025
100,000,000	0	.0000798	.0000798	.0000798	.0000798

Let $S = 51$ denote the number of States (including D.C.) and let C_k denote the number of Electoral College votes for State k, $k = 1, \ldots, S$, with $\sum_{k=1}^{S} C_k = C = 538$. Let v_k denote the number of electoral votes for candidate one from States other than k. State k's electoral votes could swing the Electoral College outcome if $C/2 - C_k \leq v_k \leq C/2 + C_k$. Let $M^{[v_k,k]}$ denote the number of subsets s of States, not including k, such that $\sum_{k' \in s} C_{k'} = v_k$. The unconditional probability that State k's electoral votes could swing the Electoral College outcome is

$$\Pr(C/2 - C_k \leq v_k \leq C/2 + C_k) = \frac{\displaystyle\sum_{v_k = C/2 - C_k}^{C/2 + C_k} M^{[v_k,k]}}{\displaystyle\sum_{v_k=0}^{C-C_k} M^{[v_k,k]}} \ .$$

That probability is tedious to evaluate. To avoid the tedious calculation here, I observe that it is of the same order of magnitude as the probability that a single voter breaks a tie in an electorate of size 50:

$$\Pr(C/2 - C_k \leq v_k \leq C/2 + C_k) \approx \Pr(50_{25}) = \binom{50}{25}/2^{50} = .11 \ .$$

That crude approximation loses one implication of the exact formula, which is the intuitively obvious point that larger States have a bigger probability of swinging the Electoral College outcome than smaller States do.

Because the population of a State is anywhere from one to two orders of magnitude smaller than the population of the United States as a whole, the probability that a person's vote is decisive for a State popular vote outcome is in a range from four to ten times larger than the direct election probability for the United States as a whole. Compare the no-poll probability in Table 1 for a population of 1,000,000, which is .0008, to the probability for a population of

100,000,000, which is .00008. To find the joint probability that one is decisive for the State outcome and one's State swings the Electoral College, we multiply the probabilities of the two events. This entails an assumption that the two events are independent. Multiplying the smaller probabilities for the State outcomes by the crudely approximated probability for the Electoral College outcome, we find that the probability of being decisive is roughly the same as it would be if the presidential election were decided by a direct popular vote.

At the crude level of calculation attempted here, having the Electoral College appears not to make much difference for the probability that an individual is decisive for the national election outcome. It might be interesting to extend such calculations, in at least two directions. (1) Rigorous calculation of the Electoral College outcome probabilities, combined with use of actual State population sizes, would show either a bias in favor of residents of small States or a bias in favor of residents of large States. (2) It would be interesting to apply similar kinds of calculations to determine the probability that a voter is decisive for party control of the House of Representatives.

For more on the probability of being decisive and the Electoral College, see the following paper.

Gelman, Andrew, Gary King and W. J. Boscardin. 1998. "Estimating the Probability of Events That Have Never Occurred: When is Your Vote Decisive?" *Journal of the American Statistical Association* 93: 1–9.