Election Fraud or Strategic Voting?*

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Abstract

I simulate a mixture process that generates individual preferences that, when aggregated into precincts, have counts whose second significant digits approximately satisfy Benford's Law. By deriving sincere, strategic and coerced votes from these preferences under a plurality voting rule, I find that tests based on the second digits of the precinct counts are sensitive to differences in how the counts are derived. The tests can distinguish coercion from strategic voting, and can even detect roll-off. With a very large number of precincts the tests may be able to distinguish strategic from nonstrategic voting. These simulation findings are supported by data from federal and state elections in the United States during the 1980s and 2000s, from the 2006 election in Mexico and from the 2009 presidential election in Iran.

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Introduction

Voting is complicated, and diagnosing whether something is wrong with the vote count in an election should take the complications into account. Among the primary complications any diagnostic scheme needs to acknowledge is strategic voting: when some voters take the preferences, beliefs and likely behavior of other voters into account, many may cast votes that differ from what they would do if they acted based solely on their own preferences. Under an assumption that most voters behave rationally, theory has been developed to describe the consequences of such strategic behavior in many circumstances. The literature bearing on this topic is evidently too vast to be summarized here, but Cox (1994, 1996) discusses the ideas and demonstrates the existence of the phenomenon of primary immediate interest. Particularly I am concerned with the so-called "wasted vote logic" according to which some voters decide to vote not for their most preferred choice but instead for a lower ranked alternative in order to try to defeat an even lower ranked alternative that they believe is attracting more votes than their first choice is attracting. Cox (1994) developed this idea in connection with his $M + 1$ rule: if there is a single nontransferable vote (SNTV) system for M offices, then Duvergerian equilibria may exist in which no more than $M + 1$ candidates receive a positive proportion of the votes.

Wasted vote logic can produce results that are surprising if one knows about voters' preferences but not about their beliefs or strategies. Some candidates may receive many more votes than preferences alone would indicate, while others surprisingly receive very small or even negligible shares of the vote. Allegations that there are irregularities in vote counts may seem plausible in such circumstances if the possibility that there was strategic voting is ignored.

Here I'm concerned with tests that purport to diagnose election irregularities in the absence of information even about preferences. Whether such diagnosis is possible at all is of course a question, but the evidence so far is pretty strong that some preference-free diagnostic methods can detect problems (Pericchi and Torres 2004; Mebane 2006, 2007b,

2008; Mebane and Kalinin 2009; Mebane 2010). The referent tests don't use any information about preferences, but instead look at patterns in the second significant digits of low-level (say precinct) vote counts. If the distribution of those digits differs significantly from the one implied by Benford's Law, then supposedly there is something wrong with the election; at least, investigation using much richer kinds of information is warranted. The issue here is whether this kind of test can distinguish irregularities from strategic voting. To put it a little more sharply, can the tests distingush election fraud from normal politics?

Even though the tests proceed without having any information about preferences at all, a conceptual challenge expressed in terms of preferences may help to frame the issue in a clear way. Imagine two different scenarios for election day. In one, a voter arrives at the polls to find there a big man with a gun, who tells the voter the voter must vote the way he says or else he will return after the election and kill the voter's family and burn down the voter's village. Since the voter surmises that every other voter at that polling place is being similarly threatened, the voter complies and votes as instructed, different from the way the voter originally planned to vote. In the other scenario, there is no man with a gun, but while traveling to vote the voter hears a credible news report stating that preelection surveys suggest the election is very close between the top two parties, with the voter's most preferred party coming in a distant third. The voter decides to abandon his most preferred party and instead vote for the one of the top two parties that he likes the most.

In both scenarios, the voter's choice is determined not by the voter's own preferences but by someone else's preferences. One might argue that having one's vote determined by someone else is the core element of election fraud (cf. Lehoucq 2003). In the first scenario, the preferences represented by the man with a gun rule. No matter what the voter may think about the election, electorally irrelevant considerations such as not wanting his family murdered override what the voter was otherwise planning to do. In the second scenario, there is no coercion, but the voter responds to other voters' preferences and changes his vote. The voter's electorally relevant preferences play some role—citing Cox's

theory we may assume the voter does not vote for his least preferred party—but still his choice depends on someone else's desires. But only the first scenario represents fraud.

Can tests based on the second significant digits of vote counts distinguish the man with a gun from normal strategic voting? This paper takes up this question. For motivation there is the general conceptual puzzle just considered, but there is also a specific empirical challenge. Mebane (2008) concluded that "as measured by the [second-digit Benford's Law (2BL)] test, signs of election fraud in recent American presidential votes seem to be rare." As I will demonstrate below, this impression appears to be erroneous. A different form of test than was used in Mebane (2008) shows extensive and significant departures from the 2BL pattern in American elections during both the 1980s and the 2000s. The departures affect not only votes recorded for president but for other federal offices such as the U.S. House of Representatives and the U.S. Senate. Election returns for state-level offices, such as votes for state legislative seats and for governor, similarly fail to follow the basic 2BL distribution. The patterns of departure from 2BL are similar across all these offices.

Since widespread fraud reaching across thousands of election contests and over several decades in the United States is not a likely possibility, I investigate whether another explanation holds, particularly the effect that strategic voting has on 2BL tests. The answer is that it does. The same line of thinking also explains some findings bearing on the 2006 election in Mexico (Mebane 2007a). Finally, reconsidering some results regarding the presidential election of 2009 in Iran, reported in Mebane (2010), shows that even with strategic voting taken into account, 2BL tests still should not be used thoughtlessly.

After reviewing some basic definitions for 2BL tests, I start with a Monte Carlo simulation study that illustrates the different effects strategic voting and coercion have on the distribution of second digits in vote counts. Then I examine data from the aforementioned elections. I conclude with some thoughts about how the kind of natural history exercise conducted in this paper may ultimately be supported by deductive theory, and about why such theory is some ways off.

Second-digit Benford's Law (2BL) Tests

Benford's Law describes a distribution of digits in numbers that arises under a wide variety of conditions. Statistical distributions with long tails (like the log-normal) or that arise as mixtures of distributions have values with digits that often satisfy Benford's Law (Hill 1995; Janvresse and de la Rue 2004; Rodriguez 2004). Under Benford's Law, the relative frequency of each second significant digit $j = 0, 1, 2, \ldots, 9$ in a set of numbers is given by $r_j = \sum_{k=1}^{9} \log_{10}(1 + (10k + j)^{-1})$. Benford's Law has been used to look for fraud in finance data (Cho and Gaines 2007). In general the digits in vote counts do not follow Benford's Law, but several examinations have found Benford's Law often approximately describes vote counts' second digits (e.g. Mebane 2006).

Tests of whether the second digits of vote counts are distrbuted according to Benford's Law come in two forms. One uses a Pearson chi-squared statistic: $X^2_{2BL} = \sum_{j=0}^{9}(n_j - Nr_j)^2/(Nr_j)$, where N is the number of vote counts of 10 or greater (so there is a second digit), n_j is the number having second digit j and r_j is given by the preceding formula as $(r_0, \ldots, r_9) = (.120, .114, .109, .104, .100, .097, .093, .090, .088, .085)$. If the counts whose digits are being tested are statistically independent, then this statistic should be compared to the chi-squared distribution with nine degrees of freedom.

The second form of test, inspired by Grendar, Judge, and Schechter (2007), considers the mean of the second digits. If the counts' second-digits follow Benford's Law, then the value expected for the second-digit mean is $\bar{j} = \sum_{j=0}^{9} jr_j = 4.187$. I use $\hat{j}$ to denote the estimated second-digit mean.

Simulating Strategic Voting and Coercion

I simulate an SNTV election for $M = 1$ office—a simple plurality election—based on artificial preferences generated so that nonstrategic votes approximately satisfy 2BL. For realism, to match in particular the findings of Mebane (2006), the first significant digits of the artificial votes do not satisfy Benford's Law. Then I simulate the effects both of

strategic voting, where voters who most prefer a losing candidate switch their votes to one of the top two finishers, and coercion, where some voters vote for a candidate regardless of their preferences. The simulation is constructed as a Monte Carlo exercise, so results reflect the average from hypothetically rerunning the election under the same conditions many times. In real data such repetitions do not occur, of course, but often the repeated sampling methodology is invoked to support studying observed statistics.

I simulate and then count votes by individuals in a set of precincts. Benford's Law distributions of digits are known to arise from processes that are mixtures of distributions, and Mebane (2006) shows the same kind of origin for counts that satisfy 2BL but not Benford's Law. Mebane (2006) and Mebane (2007c) simulate precinct data that satisfy 2BL, and the approach taken here is prompted by ideas used in those simulations.

The idea is to simulate precincts that contain individuals who have preferences for each of four candidates, preferences generated from a set of mixture distributions. It may help to think of precincts as having different concentrations of more or less intense partisans, even though of course there is no real political content to the numbers used in the simulation.

Each precinct has a basic offset selected using a uniform distribution on the interval $[-2, 2]$: $\mu \sim U(-2, 2)$. This determines the average "partisanship" of voters in the precinct. There is a randomly generated number of voters in each precinct who have similarly generated preferences. Let $m_0 \sim P(\mathcal{M})$ denote an initial value for the number of eligible voters in the precinct, based on the Poisson distribution with mean $\mathcal{M}$. In the current simulation, $\mathcal{M} = 1300$. The number of different types of eligible voters in the precinct is an integer $K \sim I(2, 15)$ chosen at random with probability 1/14 from the set $\{2, \ldots, 15\}$. The number of eligible voters of each type is a Poisson random variable $m_i \sim P(m_0/K)$, $i = 1, \ldots, K$. Hence the total number of eligible voters in the precinct is $\tilde{m} = \sum_{i=1}^{K} m_i$, and the proportion of eligible voters of type i is $\phi_i = m_i/\tilde{m}$. Each voter has a preference for each candidate that depends on the voter's type. The proportions ϕ_i are used to distribute the preferences types around the precinct offset μ. Let $\bar{\phi} = K^{-1} \sum_{i=1}^{K} \phi_i = K^{-1}$ denote the

mean type set proportion. Using the normal distribution with mean zero and variance σ, denoted $N(0, \sigma)$, define $\nu_{ji} \sim N(0, \sigma\sqrt{10})$ and generate base values for the preferences of the eligible voters of type i by

$$\mu_{1i} = \mu + (\phi_i - \bar{\phi})\nu_{1i} \tag{1a}$$

$$\mu_{2i} = -\mu_{1i} \tag{1b}$$

$$\mu_{3i} = -0.1 + \mu + (\phi_i - \bar{\phi})\nu_{3i} \tag{1c}$$

$$\mu_{4i} = -0.2 + \mu + (\phi_i - \bar{\phi})\nu_{4i} \tag{1d}$$

Each normal variate is selected independently for each j and i. Hence, for example, the base value of preferences for candidate 1 held by eligible voters of type i is distributed normally with mean μ and variance $10\sigma^2(\phi_i - \bar{\phi})^2$. The average base value for preferences among all eligible voters in the precinct is μ. If μ represents the basic "partisanship" of each precinct, then the $(\phi_i - \bar{\phi})\nu_{ji}$ values represent effects different issues, performance judgments, social positions, campaign strategies and whatnot have on sets of voters.

A more positive number indicates a candidate is more preferred. Candidates 1 and 2 come from opposite "parties," while candidates 3 and 4 are typically (but not always) positioned with values slightly more negative than the values assigned to candidate 1. This structure implies that when candidate 1 is preferred to candidate 2 (i.e., when $\mu_{1i} > 0 > \mu_{2i}$), candidates 3 or 4 have some chance to be the most preferred candidate, but when $\mu_{2i} > 0 > \mu_{1i}$ candidates 3 and 4 are much less likely to be preferred over candidate 2. One might think of this as a situation in which there are two candidates that are ideologically similar to candidate 1 but usually less preferred than candidate 1.

To get preferences for individuals, I add a type 1 extreme value (Gumbel) distributed component to each individual's base preference value. Let $G(0, 1)$ denote a type 1 extreme value variate with mode 0 and spread 1. For candidate $j \in \{1, 2, 3, 4\}$, each of the m_i individuals k of type i has preference $z_{jik} = \mu_{ji} + \epsilon_{jik}$, $\epsilon_{jik} \sim G(0, 1)$, with the extreme

value variates being chosen independently for each candidate and individual. Hence each voter in the simulation has the same error structure for its preference as is implied by using a simple multinomial logit choice model (McFadden 1973).

To define the baseline of votes that are cast in the absence of strategic considerations, I define variables that measure for each individual which candidate is the first choice. This is the candidate for which the individual has the highest preference value. A qualification is that an individual does not vote unless the preferred candidate's value exceeds a threshold v. This represents the idea that not every eligible voter votes, perhaps due to the cost of voting.[1] Only candidates 1, 2 and 3 actually run, and all voters with a first-place preference for candidate 4 are coerced to vote for candidate 1 regardless of their other preferences. So for each candidate j, indicator variable y_{jik} is defined to be 1 if all the inequalities in the corresponding one of the following definitions are true, zero otherwise:[2]

$$y_{1ik} = z_{1ik} > v \wedge z_{1ik} > z_{2ik} \wedge z_{1ik} > z_{3ik} \wedge z_{1ik} > z_{4ik} \tag{2a}$$

$$y_{2ik} = z_{2ik} > v \wedge z_{2ik} > z_{1ik} \wedge z_{2ik} > z_{3ik} \wedge z_{2ik} > z_{4ik} \tag{2b}$$

$$y_{3ik} = z_{3ik} > v \wedge z_{3ik} > z_{1ik} \wedge z_{3ik} > z_{2ik} \wedge z_{3ik} > z_{4ik} \tag{2c}$$

$$y_{4ik} = z_{4ik} > v \wedge z_{4ik} > z_{1ik} \wedge z_{4ik} > z_{2ik} \wedge z_{4ik} > z_{3ik} \tag{2d}$$

Either zero or one of the y_{jik} values for each individual k will be nonzero. The total of these would-be votes for each candidate j is the sum of the y_{jik} values: $y_j = \sum_i \sum_k y_{jik}$.

The votes for candidates 1, 2 and 3 are subject to wasted vote logic. I choose σ in equations (1a)–(1d) so that candidate 3 almost always has the smallest number of first-place finishes among candidates 1, 2 and 3. Hence some voters strategically abandon candidate 3 and vote for either candidate 1 or 2. The number of switches depends on both the relative valuations of the candidates and on whether the differences between candidates

[1] From the perspective of theory about Benford's Law, using v guarantees that all voters' preferences come from the upper tails of their preference distributions.

[2] $\wedge$ denotes logical 'and'.

exceeds a threshold t: someone votes for their second-ranked candidate when their first-ranked candidate comes in last and the gaps between their choices are sufficiently large. Given that candidate 3 comes in last, the number of switched votes is

$$o_{312} = \sum_i \sum_k (z_{3ik} > v \wedge z_{3ik} > z_{1ik} + t \wedge z_{1ik} > z_{2ik} + t \wedge z_{3ik} > z_{4ik})$$

$$o_{321} = \sum_i \sum_k (z_{3ik} > v \wedge z_{3ik} > z_{2ik} + t \wedge z_{2ik} > z_{1ik} + t \wedge z_{3ik} > z_{4ik})$$

The votes for each candidate after the strategic switching to second-ranked candidates are

$$w_1 = y_1 + o_{312} \tag{3a}$$

$$w_2 = y_2 + o_{321} \tag{3b}$$

$$w_3 = y_3 - (o_{312} + o_{321}) \tag{3c}$$

Notice that if $t = 0$, then $w_3 = 0$ and candidate 3 receives no votes.

Because voters who place candidate 4 first are coerced to vote for candidate 1, the total of votes for candidate 1 is $\tilde{w}_1 = w_1 + y_4$.

Setting $\mathcal{M} = 1300$, I generate 5,000 precincts according to the preceding specifications. $\mathcal{M} = 1300$ thus becomes the average precinct size in terms of eligible voters. I replicate this simulation 500 times. Table 1 reports the mean over these replications of χ^2_{2BL}, $\hat{j}$, the standard error of $\hat{j}$ and the total number of would-be votes in y and votes in w and $\tilde{w}$.

*** Table 1 about here ***

The results suggest the pattern of second digits is sensitive to all the manipulations implemented in the simulation. First, looking at the statistics for the would-be votes y_j, χ^2_{2BL} for y_1 shows no significant departure from the 2BL pattern, while $\hat{j}$ is slightly more than two standard errors greater than $\bar{j}$: $4.29 - 2(.04) > \bar{j}$. This excess above $\bar{j}$ is caused by the presence of the two other candidates, 3 and 4, competing for first place when μ_{1i} is positive. This is evident upon contrasting the statistics for y_2. Except for the presence of

candidates 3 and 4, the preferences underlying y_2 are symmetrically opposite those underlying y_1. Solely due to the symmetry in the preference distribution, the statistics should be the same. Yet while χ^2_{2BL} again shows no significant departure from the 2BL pattern, $\hat{j} = 4.15$ for y_2 is less than but not significantly different from $\bar{j}$.[3] Considered on their own, the counts of would-be votes for candidates 3 and 4 do not have significantly discrepant χ^2_{2BL} values but do have $\hat{j}$ values significantly greater than $\bar{j}$.

Once wasted-vote logic is used to shift some votes away from candidate 3 and to candidates 1 and 2, the distribution of second digits changes noticeably. For w_1 and w_2, χ^2_{2BL} shows no significant departure from 2BL, but $\hat{j}$ is significantly greater than $\bar{j}$. These mean statistics however remain significantly smaller than the value of 4.5 that would occur if the second digits were distributed with equal frequencies (meaning, if each occurred with probability 1/10). For w_3, χ^2_{2BL} is very significantly different from what 2BL would imply, and $\hat{j}$ is substantially less than $\bar{j}$. Of course, having set $t = 0$ would have reduced w_3 to exactly zero, but setting other small values for t produces similar results.[4]

Finally, the effect of coercion is evident in the statistics for $\tilde{w}_1$. χ^2_{2BL} is very significantly different from what 2BL would imply, and $\hat{j}$ is substantially less than $\bar{j}$. Notably $\hat{j}$ here is significantly greater than $\hat{j}$ for the candidate that was abandoned for strategic reasons. The vote counts differ for the candidates, however—candidate 1 has more than 35 times the vote of candidate 3—so there should be little possibility of confusion between candidates whose statistics differ because of these respective mechanisms.

Most important is that the statistics for $\tilde{w}_1$ differ substantially from those for w_1 or even y_1. By examining the second digits of vote counts of winning candidates, fraud done by coercion seems to be distinguishable from either strategic or nonstrategic normal politics.

Distinguishing strategic from nonstrategic normal politics is a less of a sure bet. χ^2_{2BL} seems not to be useful for this purpose at all, but $\hat{j}$ does tell us something. The mean statistic for y_2 differs significantly from that for w_2, but the difference between the

[3] Here I use "significantly different" to refer to means that differ by more than two standard errors.

[4] I found similar results for all the statistics reported here for $t \in \{.5, .45, .4, .35, .3, .25, .2, .15, .1, .05, .025\}$.

statistics for y_1 and w_1 falls a bit short of statistical significance. Increasing the number of precincts to 15,000 or more would shrink the standard error of the mean and consequently produce a significant difference. Hence we might surmise that with a sufficiently large number of precincts, $\hat{j}$ can distinguish between situations where a candidate has no ideologically (or more generally, preferentially) similar competition due to voters having strategically abandoned all such candidates from the situation where such candidates never existed. The latter case might arise, for instance, where elites or processes (say primaries) act to keep the other candidates off the ballot and out of voters' considerations. A much larger number of precincts seem to be required to distinguish wasted-vote strategic voting from the situation where similar but less preferred candidates appear on the ballot in the absence of strategic voting. In both of these latter cases, significant deviations from 2BL in $\hat{j}$ can occur, but the mean appears to be slightly larger when there is strategic voting.

Whether these interpretations of 2BL test statistics are warranted in real data of course depends on how well the conditions of the simulation capture the circumstances in real elections. The simulation, while perhaps complicated, is not particularly realistic. Precinct sizes, for instance, do not generally follow a Poisson distribution.[5] Specifying that each precinct contains between two and fifteen different types of voters also seems unnecessarily restrictive. Other features of the simulation also are admittedly artificial.[6] The least one can say is that real data represent mixtures that are much more complicated and irregular than the simulation. Rather than attempt to make the simulation much more realistic at this point, I turn instead to real data from some actual elections.

[5]Nor do precinct sizes follow a negative binomial distribution as was used in the "calibration" effort of Mebane (2007c).

[6]The simulation results themselves are stable within a range of variation of the model conditions. Using $v = 2$ produced similar results, but using $v = 1.5$ produced departures from 2BL in y_2 that were detectable by χ^2_{2BL}. For $\mathcal{M} \in \{1200, 1400, 1500\}$, $\hat{j}$ for y_2 remains not significantly different from $\bar{j}$, so that the other statistics can be considered relevant. In these cases the statistics for the other vote totals behave as described in the text. For $\mathcal{M} \in \{800, 900, 1000, 1100\}$, $\hat{j}$ for y_2 differs significantly from $\bar{j}$.

Recent Elections in the United States

I consider precinct data from several elections conducted in the United States of America during the 1980s, 1990 and the 2000s.[7] For several years, I have vote totals reported for both federal and state offices.[8] For the 1980s and 1990 the data include every state except California. For the other years data were obtained for most but not all states (including DC): 36 states in 2000; 44 states in 2004; 33 states in 2006; 40 states in 2008. Data are not available for every precinct in some states.

To come quickly to the kind of pattern that typifies these data, consider the display based on votes recorded for president in 2008, shown in Figure 1. $\hat{j}$ is shown separately in four categories. Clockwise from the upper left in the display these are means for the Republican candidate in states where the Republican won, for the Republican candidate in states where the Democrat won, for the Democratic candidate in states where the Democrat won and for the Democratic candidate in states where the Republican won. States are placed along the x-axis at locations corresponding to the absolute margin between the Democratic and Republican candidates in each state.[9] Each plot shows a nonparametric regression curve (Bowman and Azzalini 1997) that indicates how the mean of the second digit of the vote counts for the candidate in each category varies with the state absolute margin. Use $\hat{j}_x$ to denote this conditional mean. $\hat{j}_x$ is shown surrounded by 95 percent confidence bounds. The question is whether $\bar{j}$, indicated by a horizontal dotted line in the plots, falls outside of the confidence bounds. In such cases I say $\hat{j}_x$ differs significantly from $\bar{j}$.

*** Figure 1 about here ***

[7]The 1980s and 1990 data come from the Record of American Democracy (King, Palmquist, Adams, Altman, Benoit, Gay, Lewis, Mayer, and Reinhardt 1997). Data from 2000 and 2004 come from the Atlas of U.S. Presidential Elections (Leip 2004) and from collections done by the author. Data from 2006 and 2008 were collected by the author.

[8]I have data for federal and state elections for the 1980s, 1990, 2006 and 2008. For 2000 and 2004 I have only presidential election data.

[9]The absolute margin is the absolute difference between vote proportions.

I use the margin to array the means because of the idea that closer races may prompt more strategic voting according to wasted vote logic. Given the Electoral College, the outcome in each state does not depend on the votes in any other state. So if voters think of each state as a separate contest, one might expect to see more evidence of strategic voting in so-called battleground states than in states where voting is more lopsided. Neither the results shown in Figure 1 nor information to be shown below support the idea that voters think of their votes for president as being involved in separate state contests. In presidential votes, wasted vote logic appears to apply across the board.

In light of the simulation results, the evidence in favor of strategic voting in Figure 1 is that for the Democrat's votes in states where the Democrat won, $\hat{j}_x$ is about 4.3 throughout most of the distribution. The confidence bounds are always greater than $\bar{j}$ and rise to include 4.5 only at the top of the distribution of state margins. Hence $\hat{j}_x$ is fairly precisely in line with the mean associated with the winning candidate under strategic voting in the simulation. $\hat{j}_x$ is also compatible with the mean under nonstrategic voting, but evidence to be considered shortly argues strongly that strategic voting does occur.

In contrast, the confidence bounds for $\hat{j}_x$ for either the Republican where he lost or for the Democrat where he lost usually include $\bar{j}$. Based on the simulation, the interpretation would be that the candidates in these races received relatively few votes due to wasted-vote strategic voting. The confidence bounds include $\bar{j}$ for the Republican candidate in states the Republican won over about 60 percent of the distribution but otherwise are greater than $\bar{j}$. Using the simulation results, this might indicate that the Republican received some strategic votes in these states, though perhaps a smaller proportion than the Democrat received in states where the Democrat won.

The additional evidence in favor of strategic voting comes from a test of part of the "bimodality" hypothesis introduced in Cox (1994): if there is a Duvergerian equilibrium so that the $M + 1$ rule holds, then the ratio of the second loser's vote total to the first loser's vote total should be approximately zero. Table 2 shows that in 2008 this relationship holds

in all states for which I have data: I compute the referent ratio in each state, with the rank order of candidate finishes being determined separately for each state.[10] The biggest ratios are observed in Illinois (.068), DC (.066) and Montana (.046). These politically negligible values are not substantially different from zero.

*** Table 2 about here ***

The pattern of $\hat{j}_x$ values being significantly greater than $\bar{j}$ but less than 4.5 for several candidates is apparent among votes for president in other years. Consider, for example Figure 2, which shows the same kind of plots shown in Figure 1 except now for 1984. Again values near 4.3 are evident for the Republican candidate in states where the Republican won. $\hat{j}_x$ is significantly greater than $\bar{j}$ for the Democrat in states where the Republican won, over about half of the distribution. States where the Democrat won are too few to allow $\hat{j}_x$ to be estimated reliably. Or consider Figures 3, 4 and 5. Figure 3 shows that in 1988 $\hat{j}_x > \bar{j}$ over most of the distribution for both candidates in states where the Republican won but not in states where the Democrat won. Figure 4 shows that in 2000 $\hat{j}_x > \bar{j}$ for the Democrat in states where the Democrat won but mostly not for the other three categories. In this case a higher proportion of votes going to the Democrat than of votes going to the Republican appears to have been strategic. Figure 5 shows that in 2004 $\hat{j}_x > \bar{j}$ again for the Democrat in states where the Democrat won but also for some portions of the distribution for the other three categories.

*** Figures 2, 3, 4 and 5 about here ***

When considering other federal offices, similar patterns are observed but also there is a new phenomenon. For legislative elections, the x-axis in plots of $\hat{j}_x$ now contains the absolute margins in legislative districts. Figure 6 shows analysis done on the data for U.S. House elections pooled over years 1984, 1986, 1988 and 1990. $\hat{j}_x$ shows how the mean of the second digit of the vote counts for the respective candidates in each category varies with

[10]The ratios are computed between the state vote totals for candidates by name, hence separate party totals are summed to compensate for fusion in New York. In Oklahoma the data showed no votes for any candidate other than the Democrat and the Republican; hence the ratio is exactly zero for Oklahoma.

the district absolute margin. For most of the distribution of both Republican and Democratic winners $\hat{j}_x > \bar{j}$, although $\hat{j}_x$ is not distinguishable from $\bar{j}$ in low-margin races where the Republican won and in high-margin races for both Republican and Democratic winners. The new pattern is that $\hat{j}_x < \bar{j}$ for much of the distribution for losing candidates. $\hat{j}_x$ in these cases falls to values as low as 4.0.

*** Figure 6 about here ***

Similar results are observed for the U.S. Senate. I again pool election data from 1984, 1986, 1988 and 1990. The results, presented in Figure 7, show a pattern in which $\hat{j}_x > \bar{j}$ over most races for Republican winners and for Democratic winners, although $\hat{j}_x$ is again not distinguishable from $\bar{j}$ in low-margin races where the Republican won, and $\hat{j}_x < \bar{j}$ in high-margin races for Democratic winners. $\hat{j}_x$ is not significantly different from $\bar{j}$ over most races for Republican losers, and $\hat{j}_x < \bar{j}$ over many races for Democratic losers.

*** Figure 7 about here ***

During the 2000s the patterns change slightly. As can be seen in Figure 8, which shows the results from U.S. House elections during 2006 and 2008, in these years $\hat{j}_x$ is not distinguishable from $\bar{j}$ for most of the distribution for Democratic losers, although $\hat{j}_x < \bar{j}$ for Republican losers in high-margin races. Now $\hat{j}_x > \bar{j}$ over the entire distribution for Democratic winners. For Republicans winners $\hat{j}_x < \bar{j}$ once again in low- and high-margin races but $\hat{j}_x > \bar{j}$ otherwise. Figure 9 shows results for 2006 and 2008 U.S. Senate elections. Now $\hat{j}_x > \bar{j}$ for most Democratic winners, but $\hat{j}_x$ is no longer distinguishable from $\bar{j}$ for Republican winners. Reversing the earlier pattern, now $\hat{j}_x < \bar{j}$ for many Republican losers and $\hat{j}_x > \bar{j}$ for many Democratic losers. For Republican losers in low-margin races and Democratic losers in high-margin races $\hat{j}_x$ is not distinguishable from $\bar{j}$.

*** Figures 8 and 9 about here ***

Patterns for state legislative election data resemble those observed for U.S. House elections. Figure 10, which shows $\hat{j}_x$ for state house and state senate data pooled over years 1984, 1986, 1988 and 1990, strongly resemble Figure 6. These patterns change in the 2000s. Figure 11 shows $\hat{j}_x$ for 2006 and 2008 state house and state senate elections. In these elections Democratic winners always have $\hat{j}_x > \bar{j}$ while $\hat{j}_x$ is mostly indistinguishable from $\bar{j}$ for Republican winners. Both Republican losers and Democratic losers often have $\hat{j}_x < \bar{j}$.

*** Figures 10 and 11 about here ***

Patterns for gubernatorial election data somewhat resemble the patterns observed for U.S. Senate elections for the same years. $\hat{j}_x$ for these elections shows how the mean of the second digit of the vote counts for the respective candidates in each category varies with the absolute margin in each state. In Figure 12, which presents results for gubernatorial elections pooled over years 1984, 1986, 1988 and 1990, we see that $\hat{j}_x > \bar{j}$ for Republicans in close elections, $\hat{j}_x < \bar{j}$ for a range of somewhat close elections, and $\hat{j}_x$ is not distinguishable from $\bar{j}$ otherwise. The same pattern holds for Democrats except the degree to which $\hat{j}_x < \bar{j}$ for a range of elections is more pronounced. Figure 13 shows that by 2006 and 2008 the patterns change. Now $\hat{j}_x$ is not distinguishable from $\bar{j}$ for Republicans except for a small range of Republican winners for whom $\hat{j}_x < \bar{j}$. The pattern of $\hat{j}_x > \bar{j}$ is somewhat more prevalent than it was previously for Democrats, even though for Democratic losers there remains a range of elections in which $\hat{j}_x < \bar{j}$.

*** Figures 12 and 13 about here ***

Overall it seems that presidential elections are like other federal and state elections in one major respect: winners often attract many strategic votes, and these votes cause $\hat{j}_x$ to be significantly greater than $\bar{j}$—indeed, $\hat{j}_x$ is very often near the $\hat{j}$ values associated with wasted-vote logic switching in the simulation. Losers sometimes attract strategic vote switchers, but less frequently than do winners. Exceptions are the 2000 and 2004

presidential elections.[11] Evidence from Cox's bimodality test is very strong that strategic voting in line with wasted-vote logic is occurring in recent presidential elections, so it seems reasonable to conclude that such strategic voting is occurring in the other elections and that the second-digit means are being affected by that behavior. The statistics' significant deviations from 2BL in this sense reflect normal politics.

But a phenomenon that is observed in the other elections' vote totals but not in presidential elections is that often $\hat{j}_x < \bar{j}$. In the simulation such a second-digit mean is observed for the candidate that is strategically abandoned and in the case of coercion. But here $\hat{j}_x < \bar{j}$ is occcuring for first losers, so strategic abandonment is out of the question. And occasionally $\hat{j}_x < \bar{j}$ for winners, e.g., for U.S. Representatives, U.S. Senators and state senators during the 1980s and for governors. What can explain this pattern?

An obvious possible explanation is roll-off. Roll-off occurs when some voters vote for the office listed at the top of the ballot but not for subsequent offices. The reasons for roll-off have been debated for many years (Burnham 1965; Walker 1966; Rusk 1970), but by now it is established that roll-off is affected by many factors including voter knowledge, voting machine technology and ballot formats, and campaign and candidate characteristics (Darcy and Schneider 1989; Nichols and Strizek 1995; Wattenberg, McAllister, and Salvanto 2000; Herron and Sekhon 2005).

That roll-off is a plausible mechanism to explain why $\hat{j}_x < \bar{j}$ is supported by results from a slight modification to the simulation. Suppose now voters who support a losing candidate are less likely to vote as the margin of defeat is larger. There are many ways to implement such an idea. I proceed by assuming that somehow there is good preelection information about intended election strategies, hence roll-off is a function of would-be strategic votes. Define a logistic function of the ratio between votes for the first- and second-place candidates as follows: $f = 2\,/(1 + \exp\left[b\left(1 - w_1/w_2\right)\right])$ If $w_1 = w_2$, then $f = 1$, but given $b > 0$ then $w_1 > w_2$ implies $f > 1$. I use f to modify the turnout thresholds in

[11] Of course, who won the overall election in 2000 is controversial (Wand, Shotts, Sekhon, Mebane, Herron, and Brady 2001; Mebane 2004).

the vote-counting rules for candidate 2. The modified nonstrategic votes are $y^*_{2ik} = z_{2ik} > fv \wedge z_{2ik} > z_{1ik} \wedge z_{2ik} > z_{3ik} \wedge z_{4ik} < \max_{j=1}^{3}(z_{jik})$. As the gap between the votes for candidates 1 and 2 increases, an eligible voter who prefers candidate 2 has to have increasingly intense preferences in order to motivate actually voting. The total of these votes is $y^*_2 = \sum_i \sum_k y^*_{2ik}$. The number of strategic switchers who vote for candidate 2 is similarly modified: $o^*_{321} = \sum_i \sum_k (z_{3ik} > fv \wedge z_{3ik} > z_{2ik} + t \wedge z_{2ik} > z_{1ik} + t \wedge z_{3ik} > z_{4ik})$. The votes for candidate 2 after strategic switching are now $w^*_2 = y^*_2 + o^*_{321}$.

Using the same simulation settings as before and setting $b = 10$, I find that the mean value of $\hat{j}$ for y^*_2 is 4.04 with a mean standard error of 0.074, while the mean value of $\hat{j}$ for w^*_2 is 4.31 with a mean standard error of 0.043.[12] With the simulated roll-off, $\bar{j}$ is just less than two standard errors greater than $\hat{j}$ for y^*_2: $4.04 + 2(.074) = 4.188$. Roll-off has no effect on $\hat{j}$ for the strategic vote total w^*_2. So roll-off could explain the patterns seen in real data for losing candidates, if we also conclude that losing candidates in those cases attract a negligible proportion of strategic votes.

I conjecture that a similar effect could be produced for some winners in cases where the race is so lopsided that even some who support the winner decide not to turn out. The cases where $\hat{j}_x < \bar{j}$ for winners mostly occur for races with large margins of victory.

An interesting aspect of a conclusion that digit tests can detect strategic voting and roll-off is that in most of the races where there is this evidence of strategic voting, only at most two candidates appear on the ballot. In presidential races there are almost always third-party candidates, but not in most legislative contests. Some voters appear to be acting as if there are more candidates than there are. Perhaps this indicates that the possibility of writing in a candidate's name who is not on the ballot—perhaps themselves—has some effect, in that voters have to decide actively not to do that (Tullock 1975). Or it may be that other kinds of strategic consideration besides wasted vote logic are affecting votes' digits, say strategies associated with split-ticket voting and divided

[12] Similar results are obtained using $b \in \{4, 6, 8, 10, 12\}$.

government (Fiorina 1992; Alesina and Rosenthal 1995).

Computing $\hat{j}_x$ relative to the margin in each U.S. race reveals that one consequence of strategic voting is to produce election results that are not particularly close. Moreover there is a difference between the parties. At least during the 1980s, Republican winners in the U.S. House and Senate typically enjoyed larger margins where strategic voting was at its peak than did Democratic winners: $\hat{j}_x$ peaks at margins of about .45 and .3 for Republicans in contrast to about .2 and .17 for Democrats. The difference between the parties is also evident in state House and Senate races: $\hat{j}_x$ peaks at a margin of about .38 for Republican winners and about .3 for Democratic winners.

In the 2000s all these differences change such that Republicans generally receive fewer strategically switched votes while Democrats receive more. In the 2000s $\hat{j}_x$ is often not very different or not at all different from $\bar{j}$ for Republican winners, while $\hat{j}_x > \bar{j}$ for Democratic winners more often and more in races that remain close even after votes have been strategically switched. Democratic candidates in presidential races in the 2000s also evidently received higher proportions of strategically switched votes than did Republicans. These patterns of change over time are compatible with an impression that a more diverse set of voters supported Republican winners in the 1980s than did so in the 2000s.

Mexico 2006

The 2006 Mexican presidential election was close and highly controversial (European Union 2006; Klesner 2007). Five party coalitions sought votes in the election, and the winning candidate's margin of victory was 0.56 percent. The winner was Felipe Calderón of the *Partido Acción Nacional* (PAN), and the candidate receiving the second largest number of votes was Andrés Manuel López Obrador of the *Coalición por el Bien de Todos* (PBT). Also in the election was the *Nueva Alianza* (NA) party, formed as a splinter from the longtime ruling party, the *Partido Revolucionario Institucional* (PRI). Both NA and PRI fielded candidates in the 2006 presidential election. PRI formed a coalition called *Alianza*

por México (APM) with the *Partido Verde Ecologista de México*. The fifth party presenting candidates in the 2006 election was *Alternativa Socialdemócrata y Campesina* (ASDC).

PAN and PBT filed hundreds of challenges that alleged election day irregularities (European Union 2006, 42–43). The election court did not find irregularities sufficient to change the election outcome (European Union 2006, 3). The principal losing candidate was not persuaded that he had lost fairly (Estrada and Poiré 2007; Schedler 2007). At least one allegation about the election involved possible coercion: there was a claim that NA's leadership instructed teachers' union members in the party to "vote for Nueva Alianza's candidate for senator and 'diputado,' but vote for Calderon for President" (Kelley 2006). Of course this could be no more than a focal statement intended to coordinate strategic voting. For other statistical analysis of fraud allegations in the election see López (2009).

Other opportunities to observe either coercion or strategic voting come from considering how vote counts vary in relation to the mayor of each municipality. The official spreadsheet files that report the vote counts for each casilla (ballot box) locate each casilla in one of 2,422 municipalities (*municipios*). Each municipality may contain several towns and villages, but for the municipality itself there is a government and each such government has an elected mayor. Except in a number of municipalities in the state of Oaxaca, each mayor is affiliated with either a single party or a coalition of parties.[13]

While the mayor has no official role in administering the federal election, the mayor's party coalition likely corresponds to the locally dominant partisan organization. There are many ways such organizational capacity may produce distortions in the election results. For instance, the European Union reports that at some casillas "some polling station staff members did not turn up, and had to be replaced by substitutes or voters in line" (European Union 2006, 37). There is a suggestion that in some cases replacements were not haphazard but instead were planned to make sure the polling staff were controlled by

[13]According to data from the SNIM (Instituto Nacional para el Federalismo y el Desarrollo Municipal 2006), at the time of the 2006 federal election 421 of the 570 municipalities in Oaxaca had mayors selected via an indigenous method called *uso y costumbre* that does not involve affiliation with a political party. For one discussion of this electoral method, see Eisenstadt (2007).

one party's supporters (Kelley 2006).

I merge election data from the *Instituto Federal Electoral* (IFE)[14] with municipality party-affiliation data from *Sistema Nacional de Información Municipal* (SNIM) (Instituto Nacional para el Federalismo y el Desarrollo Municipal 2006).[15] The SNIM data list a party name in cases where the municipality mayor is affiliated with a single party, but for 201 of the 2440 municipalities in the file the data indicate only that the mayor is affiliated with a coalition. The SNIM data do not indicate which parties are included in each coalition. The members of each municipality coalition are identified using information organized by CIDAC,[16] in a few instances supplemented by information from the *Instituto de Mercadotecnia y Opinion* (IMO).[17] Municipality parties align by name with three of the coalitions standing for the federal election, namely PAN, APM and PBT. In some cases members of a municipality's governing coalition align with two federal election coalitions, either PAN and PBT or APM and PBT. Table 3 shows the number of municipality coalitions of each type.

*** Table 3 about here ***

In the election there are separate vote counts for president, senator (*Senadores*) and deputy (*Diputados*). Following the point made by Mebane (2006) that the casilla is too low a level of aggregation for 2BL tests to give meaningful results, I consider each of these counts aggregated to the seccion, a small administrative unit usually containing several casillas. The Mexican legislature is elected partly using a plurality rule (*Mayoría Relativa*) in single-member districts and partly using proportional representation (*Representación*

[14]The IFE URL is `http://www.ife.org.mx`. The title for the set of spreadsheets is *las Bases de Datos de Cómputos Distritales 2006*. There is one spreadsheet each for the votes cast for *Presidente de los Estados Unidos Mexicanos*, *Diputados por los Principios de Mayoría Relativa y Representación Proporcional* and *Senadores por los Principios de Mayoría Relativa y Representación Proporcional*. The exact URL for the spreadsheets varies.

[15]The party affiliations are listed in a file named `Nueva.dbf` (size 5049961 bytes, timestamp Aug 29, 2006). Municipalities for IFE and SNIM are not strictly speaking the same administrative units, although usually the geographic borders coincide when the names are the same.

[16]URL `http://www.cidac.org/`. See the link labeled *Base de datos de elecciones locales 1980-2006*.

[17]URL `http://www.imocorp.com.mx/`. Phone calls to officials in each municipality were required to resolve contradictions among SNIM, CIDAC and IMO regarding a few municipalities in Coahuila, Chiapas, Sinaloa and Sonora.

Proporcional) within five large regional districts. I look only at the plurality results.

Table 4, which shows the number of secciones having a vote count greater than 9 for each combination of office, municipality coalition type and party choice, suggests that the number of votes for NA and ASDC candidates varies substantially for different offices. Over all kinds of municipality coalitions, NA received many fewer votes for president than for the other offices, while ASDC received fewer votes for senator or deputy than for president. The pattern for NA is in line with the instructions quoted above. Of the other parties, PAN and PBT do better in getting higher vote counts for president than for the other offices, while APM candidates mostly get more votes for senator and deputy. The exception here is the votes for APM in municipalities with Other coalitions.

*** Table 4 about here ***

Drawing on the simulation results, the $\hat{j}$ values shown in Table 5 suggest that all these patterns result from strategic voting, albeit strategic voting that depends on the municipality's partisan configuration. Focus first on the statistics for votes for president. Votes for PAN where there is a PAN mayor and votes for PBT where there is a PBT mayor have $\hat{j}$ significantly different from $\bar{j}$ but not from the $\hat{j}$ value observed for w_1 in the simulation.[18] The values for $\hat{j}$ for APM and NA are significantly less than $\bar{j}$ in municipalities with PAN or PBT mayors. The $\hat{j}$ values are not as small as $\hat{j}$ is for w_3 in the simulation, but they match values that can be produced by slightly increasing the threshold t for the size of differences in preference ratings that govern vote switching in the simulation.[19] The pattern suggests there is strategic voting in which some voters switch from APM and NA to one of the top two parties in municipalities where there are mayors representing those same two parties. Votes for ASDC, notably, have $\hat{j}$ values not significantly different from $\bar{j}$. Supporters of ASDC in these municipalities seem not to have switched their votes, and in these municipalities ASDC seems not to have received many strategically switched votes.

[18] I again use "significantly different" to refer to means that differ by more than two standard errors.

[19] If $t = .35$, then $\hat{j}$ for w_3 is 3.29. If $t = .5$, then $\hat{j}$ for w_3 is 3.56 and $\hat{j}$ for w_2 is 4.29.

*** Table 5 about here ***

The pattern of vote switching in municipalities with other mayoral coalitions differs slightly. Where there is an APM mayor or a PAN-PBT coalition, $\hat{j}$ for APM, NA and ASDC suggest switching away from those choices, but $\hat{j}$ for PAN or PBT does not differ significantly from $\bar{j}$. Where the mayoral coalition is Other, $\hat{j} > \bar{j}$ for PAN, PBT and ASDC, but only the latter two differences are significant, while for NA again significantly $\hat{j} < \bar{j}$. That votes are perhaps switched to one of the top two parties in the presidential election is not surprising, but its hard to see this for the votes going to ASDC. Probably this is a situation analogous to that for y_3 in the simulation, where the mere presence of preferentially similar alternatives on the ballot is enough to shift $\hat{j}$ away from $\bar{j}$.

The results for president in municipalities with an APM-PBT coalition present the only statistics that appear compatible with the simulation results for the case of coercion. Among such municipalities, $\hat{j}$ is about 4.0, and APM has the most secciones with vote counts greater than 9. But in fact PAN received more votes in such municipalities than did either APM or PBT: 458,508 versus 313,196 and 222,231, respectively. So more likely than coercion is that in these municipalities a noticeable but relatively small proportion of voters abstained, and it's a situation analogous to roll-off.

For senator and deputy $\hat{j}$ values again suggest there was strategic voting in favor of PAN and PBT, respectively, in municipalities where each party controlled the mayor's office. In this case the switched votes appear to have come from ASDC and maybe from PBT in races for senator in municipalities with a PAN mayor. For these offices there is no evidence of vote switching away from APM or NA where there are PAN or PBT mayors. $\hat{j}$ is significantly less than $\bar{j}$ for NA candidates with an APM mayor, and there is no indication that another class of candidates received strategically switched votes. Since $\hat{j}$ is in the vicinity of 4.0 here, the best interpretation is probably that this is a case of something like roll-off: NA-preferring voters abstain instead of voting for candidates in places where members of the party they once affiliated with hold sway (recall that NA

formed as a splinter from PRI, which is part of APM).

Among the other municipalities, PBT candidates for senate seem to have benefitted from strategic vote switching where there is a PAN-PBT or Other mayoral coalition, but otherwise there are only scattered signs of strategic vote switching. Coercion is not evident.

There are signs that roll-off occurs both in the United States and in Mexico, but in Mexico the phenomenon seems to depend more on local political factors. Indeed, in Mexico roll-off is apparent only in some races involving APM local control and votes for NA and ASDC party candidates. Such a pattern is understandable as a consequence of likely factional conflicts between the PRI and the NA party that is an offshoot from the PRI. In the U.S. roll-off occurs, when it does, in many non-presidential races that are not close, and it is much more evident in vote totals for Democratic losers than for Republican losers. Whether this reflects the different demographic composition of the two parties' coalitions is impossible to determine with the current data.

Iran 2009

The 2009 presidential election in Iran was not close according to official election returns, but there was controversy because the winning side was accused of committing a massive fraud, which led to widespread demonstrations that were brutally repressed (Tait and Borger 2009; Bahrampour 2009). Details and further data about the election appear in Mebane (2010), but the details that concern us here are in Table 6. The table shows that Mahmoud Ahmadinejad had the most votes, about 24.5 million, followed by Mir-Hossein Mousavi with just over 13 million. The two other candidates, Mehdi Karroubi and Mohsen Rezaee, had orders of magnitude fewer votes. These counts are based on the ballot box data described in more detail in Mebane (2010).

*** Table 6 about here ***

What matters here is the story told by the second-digit test statistics. X^2_{2BL} based on the vote count in ballot boxes shows significant departures from 2BL for three candidates,

Ahmadinejad, Karroubi and Rezaee, but not for Mousavi. In light of the simulation, the $\hat{j}$ values less than $\bar{j}$ for Karroubi and Rezaee might suggest that voters strategically abandoned those candidates, but the $\hat{j}$ values for Mousavi and Ahmadinejad are only just more than two standard errors larger than $\bar{j}$. These values, $\hat{j} = 4.22$, are significantly smaller than the values of about $\hat{j} = 4.35$ that emerge in the simulation for vote totals that have been augmented by adding a significant number of strategically switched votes. It may be that the number of sincere supporters of Karroubi and Rezaee is so small in the first place that adding some of them to Mousavi's or Ahmadinejad's vote totals does not substantially affect many ballot boxes' second digits.

More likely, though, we should take the significant but small departures from 2BL in the $\hat{j}$ values in Table 6 as a caution against taking that (or any) single statistic as a sufficient indicator for what is going on in an election. In this case, Mebane (2010) shows that computing $\hat{j}_x$ for Ahmadinejad's votes as a function of the proportion of invalid ballots in each ballot box reveals that there are significant departures from 2BL in more than a quarter of the ballot boxes (Mebane 2010, Figure 1): as the proportion of invalid ballots decreases, $\hat{j}_x$ rises to a value of about 4.33. This goes with a pattern in which the proportion of votes for Ahmadinejad increases substantially as there are fewer invalid ballots: "as the proportion invalid falls from the median value of about 0.0085 to zero, Ahmadinejad's share of the vote increases from an average of about 0.64 to an average of about 0.77" (Mebane 2010, 13). Because it is difficult otherwise to explain the connection to invalid ballot proportions, Mebane (2010) suggests this is evidence of extensive ballot box stuffing on behalf of Ahmadinejad. A crude calculation suggests that Ahmadinejad's vote is inflated by at least 3.5 million votes.[20] Whether this is the only kind of fraud perpetrated in the election is unknown.[21]

[20] This estimate is formed using the triangle formula, area $= \frac{1}{2}$base $\times$ height, to compute the area under the curve in Mebane (2010, Figure 2): a proportion of 0.02 invalid votes or fewer corresponds to about 90% of the ballot boxes, so $.9(38770288)(.75 - .55)/2 \approx 3.5$ million.

[21] Before the election the opposition was said to need a margin of 5,000,000 in order to overcome the regime's likely election manipulation: "Any lower margin [...] could be erased by the pro-Ahmadinejad Interior Ministry, which conducts the elections" (Matthews 2009).

Discussion

This paper begins with a mixture process that generates individual preferences that, when aggregated into precincts, approximately satisfy 2BL. By deriving sincere and then strategic and then coerced votes from these preferences, I find that tests based on the second significant digits of the precinct counts are sensitive to differences in how the counts are derived, at least in plurality elections for a single office. The tests can distinguish coercion—where votes are cast regardless of preferences—from strategic voting according to wasted vote logic, and they can even detect roll-off. To some extent the tests may be able to distinguish strategic from nonstrategic voting. These findings based on simulations are confirmed by some real data and otherwise support plausible interpretations of real data.

Using digit tests to understand the consequences of strategic voting and to diagnose possible fraud depends on the availability of suitable covariates. In the U.S. case the margin in each jurisdiction is the covariate used to identify $\hat{j}_x$. In Mexico the partisan composition of each municipal mayoral coalition provides the important conditioning information. In Iran, fraud could not have been diagnosed the same way without information about the proportions of invalid ballots in ballot boxes. Digits alone are about as minimal a foundation for drawing inferences about what happened in elections as might be imagined. If all one has are vote counts and consequently their digits, then there is no information about preferences, strategies, campaigns or anything else that one would normally use to try to understand what went on in an election. Given an appropriate covariate, tests based on vote counts' digits can do a lot to give strong suggestions about what happened.

Much remains to do. One desirable task is to replace the simulations with deductive, analytical arguments. Here I briefly offer some musings about this.[22] Deductive arguments will be difficult to construct. Berger (2005) proves that numbers that represent multidimensional dynamical systems usually satisfy Benford's Law. The exception is systems that are degenerate in a special way, notably those that have zero eigenvalues in

[22] In fact something like these ideas motivated the data gathering and then the simulation discussed above.

their linearizations (so-called "exponentially b-resonant spectra," Berger (2005, 228)). In an argument that focuses on the topology of mappings, Schofield (1983) demonstrates that the only sets of preferences for which majority rule cycles do not exist are those with severely limited dimensionality (2 or 3). Preference dimensions beyond those necessarily allow a "dense" set of cycles to exist (Schofield 1983, 702). It is in such circumstances with high-dimensional preferences, which arguably always hold in practice, that strategic voting can affect outcomes. Majority rule thus in some sense represents a degenerate mapping. Deductive theory to demonstrate when certain voting rules, preferences and election strategies imply particular values for the second digits of vote counts will probably need to draw together theories of these types.

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Table 1: Second-digit χ^2_{2BL} statistics, means, standard errors and "vote" totals

	y_1	y_2	y_3	y_4	w_1	w_2	w_3	$\tilde{w}_1$
χ^2_{2BL}	10.7	12.0	12.0	12.4	12.4	11.8	944.6	58.5
$\hat{j}$	4.29	4.15	4.32	4.32	4.35	4.35	2.68	3.75
s.e.	.040	.041	.041	.040	.041	.041	.043	.042
votes	200,679	271,160	181,534	164,277	329,750	309,847	13,777	494,026

Note: $n = 5000$ precincts. $\mathcal{M} = 1300$, $\sigma = 1$, $v = 1.75$, $t = 0.15$, 500 replications.

Table 2: Ratio of second loser to first loser in 2008 U.S. presidential votes

IL	DC	MT	AR	ID	WY	AZ	RI	IA	WV
.068	.066	.046	.031	.030	.030	.029	.029	.025	.023
CA	KS	GA	IN	OH	TX	VT	AK	WI	MI
.022	.020	.017	.016	.016	.016	.015	.014	.014	.012
NC	MD	ND	HI	SD	TN	WA	SC	DE	MN
.012	.010	.010	.011	.011	.011	.011	.008	.007	.007
NH	NM	NY	PA	VA	FL	LA	ME	CT	OH
.007	.007	.007	.007	.007	.005	.003	.001	.000	.000

Note: Ratio of second losing candidate's statewide vote total to first loser's. Loss order is determined separately for each state.

Table 3: Municipality Party Affiliations as of the Mexican 2006 Federal Election

Municipality Party Coalition Membership

	PAN	APM	PBT	PAN-PBT	APM-PBT	Other
municipalities	534	782	396	50	56	1014
secciones	17,721	19,192	10,534	1,666	2,556	13,020

Notes: Each municipality's party affiliation is determined by matching the members of the mayor's coalition to the parties and coalitions presenting candidates in the 2006 federal election. The number of municipalities is the number appearing in the IFE data. The number of secciones is the number used for voting in the presidential election.

Table 4: Mexico 2006 Federal Election: Secciones with Counts > 9 by Municipality Party

	Party	Municipality Party Coalition Membership					
	Voted	PAN	APM	PBT	PAN-PBT	APM-PBT	Other
President	PAN	17,667	18,341	9,584	1,627	2,459	12,812
	APM	17,620	19,084	10,304	1,663	2,539	12,705
	PBT	17,243	18,570	10,436	1,595	2,412	12,887
	NA	3,740	3,258	2,183	312	582	2,228
	ASDC	10,957	8,173	5,162	680	1,287	9,105
Senator	PAN	17,664	18,294	9,537	1,624	2,451	12,547
	APM	17,642	19,121	10,341	1,665	2,542	12,681
	PBT	17,025	18,229	10,425	1,566	2,374	12,569
	NA	11,941	10,293	6,508	1,111	1,846	9,751
	ASDC	7,663	5,573	4,476	250	771	7,697
Deputy	PAN	17,662	18,310	9,554	1,623	2,449	12,542
	APM	17,632	19,107	10,331	1,664	2,541	12,682
	PBT	17,012	18,297	10,419	1,565	2,364	12,575
	NA	12,429	11,459	6,626	1,056	1,794	9,869
	ASDC	8,071	5,587	4,643	236	795	7,925

Notes: N of vote counts ≥ 10. Each *casilla extraordinaria* used for presidential voting is treated as a separate seccion. The values for Senator and Deputy use only *Mayoría Relativa* vote counts.

Table 5: Mexico 2006 Federal Election: Second Digit Means by Municipality Party

	Party	Municipality Party Coalition Membership					
President	Voted	PAN	APM	PBT	PAN-PBT	APM-PBT	Other
	PAN	4.34*	4.20	4.24	4.18	4.19	4.25
	APM	4.07*	4.13*	3.99*	3.95*	3.93*	4.13
	PBT	4.16	4.20	4.36*	4.24	4.03*	4.33*
	NA	3.28*	3.29*	3.20*	2.90*	3.14*	3.49*
	ASDC	4.18	3.99*	4.19	3.63*	3.97*	4.29*
	Party	Municipality Party Coalition Membership					
Senate	Voted	PAN	APM	PBT	PAN-PBT	APM-PBT	Other
	PAN	4.31*	4.19	4.16	4.07	4.13	4.22
	APM	4.19	4.14	4.14	4.00*	4.12	3.97*
	PBT	4.13*	4.16	4.25*	4.40*	4.16	4.34*
	NA	4.14	4.11*	4.09	4.09	4.13	4.28*
	ASDC	3.82*	3.68*	4.03	2.54*	3.42*	4.24
	Party	Municipality Party Coalition Membership					
Deputy	Voted	PAN	APM	PBT	PAN-PBT	APM-PBT	Other
	PAN	4.33*	4.23	4.17	4.11	4.06	4.23
	APM	4.19	4.22	4.17	4.12	4.15	3.97*
	PBT	4.09*	4.17	4.31*	4.13	4.31	4.25
	NA	4.14	4.07*	4.12	4.17	4.17	4.25
	ASDC	3.82*	3.70*	4.05*	2.89*	3.26*	4.23

Notes: $\hat{j}$. * shows values that differ by more than two standard errors from $\bar{j}$. Tests are based on seccion vote counts greater than 9 for the referent party. Each *casilla extraordinaria* used for presidential voting is treated as a separate seccion. The statistics for Senator and Deputy use only *Mayoría Relativa* vote counts.

Table 6: Second-digit Tests, Iran 2009 Presidential Election, Ballot Box Vote Counts

Candidate	votes	N	X^2_{2BL}	$\hat{j}$	s.e.
Mousavi	13,258,464	45,435	10.1	4.22	.014
Karroubi	330,183	10,155	290.7	3.71	.028
Rezaee	656,150	19,151	145.8	3.95	.021
Ahmadinejad	24,525,491	46,344	60.9	4.22	.014

Note: N denotes the number of ballot boxes with ten or more votes for the candidate.

Figure 1: Vote Counts for President, 2008

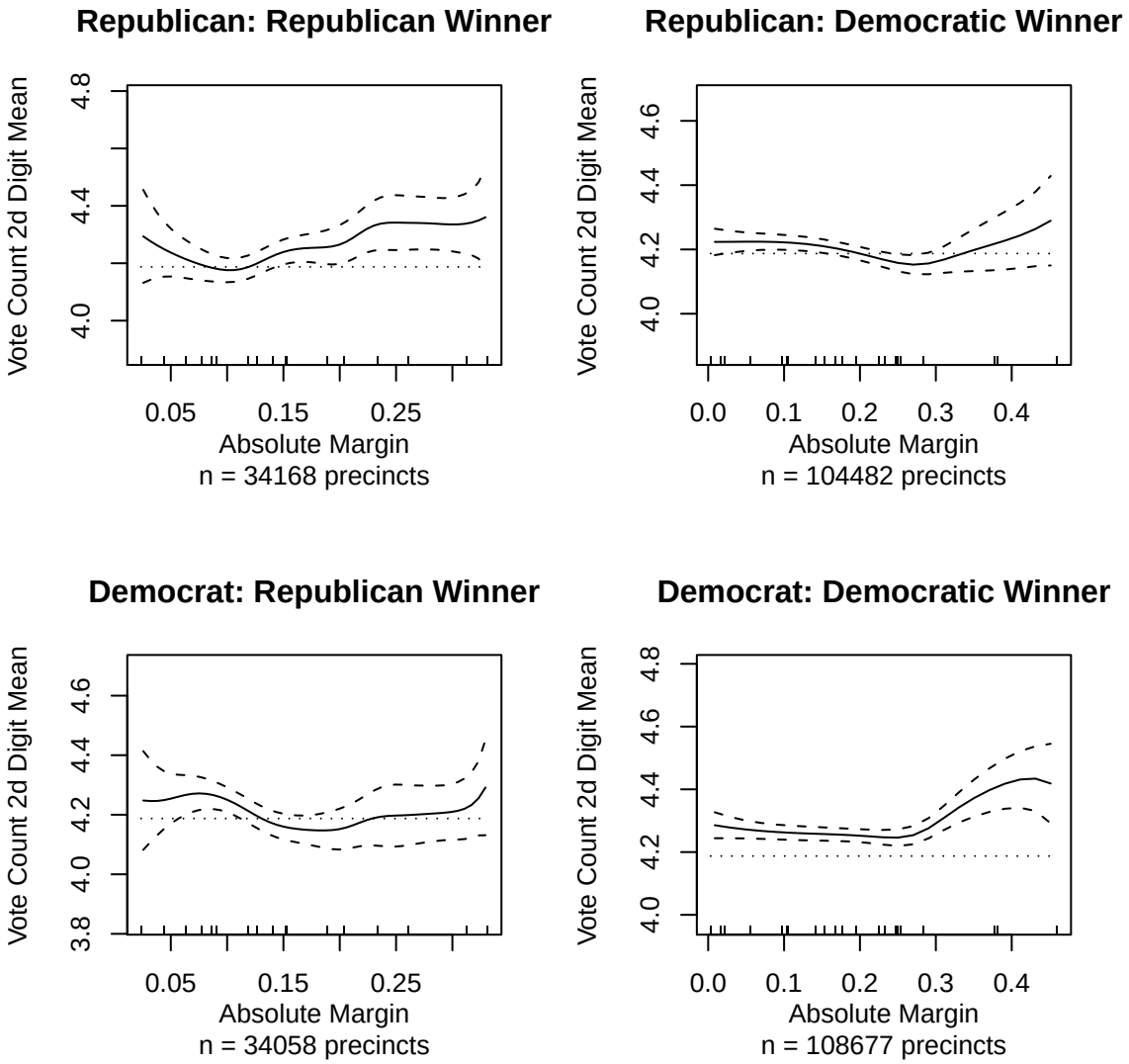

Figure 2: Vote Counts for President, 1984

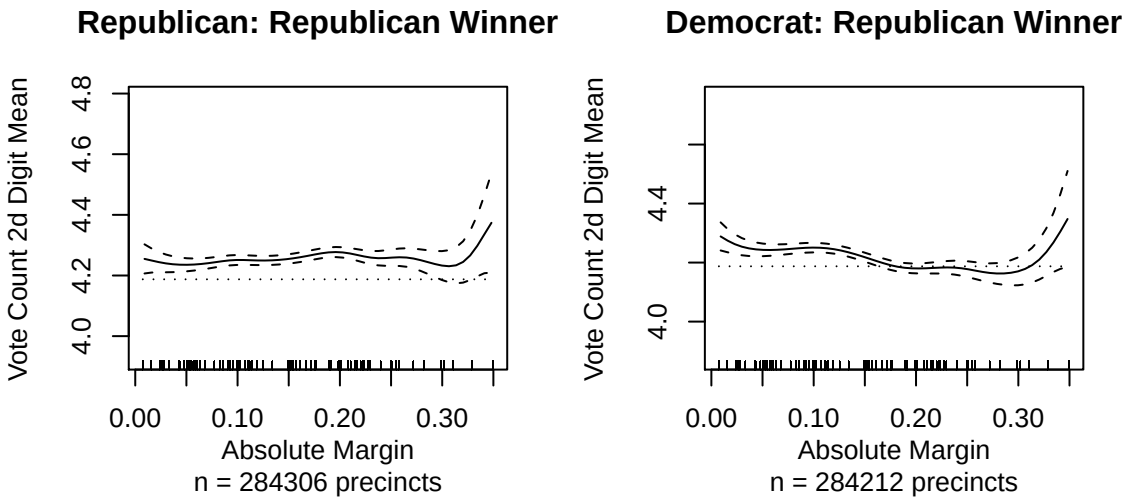

Figure 3: Vote Counts for President, 1988

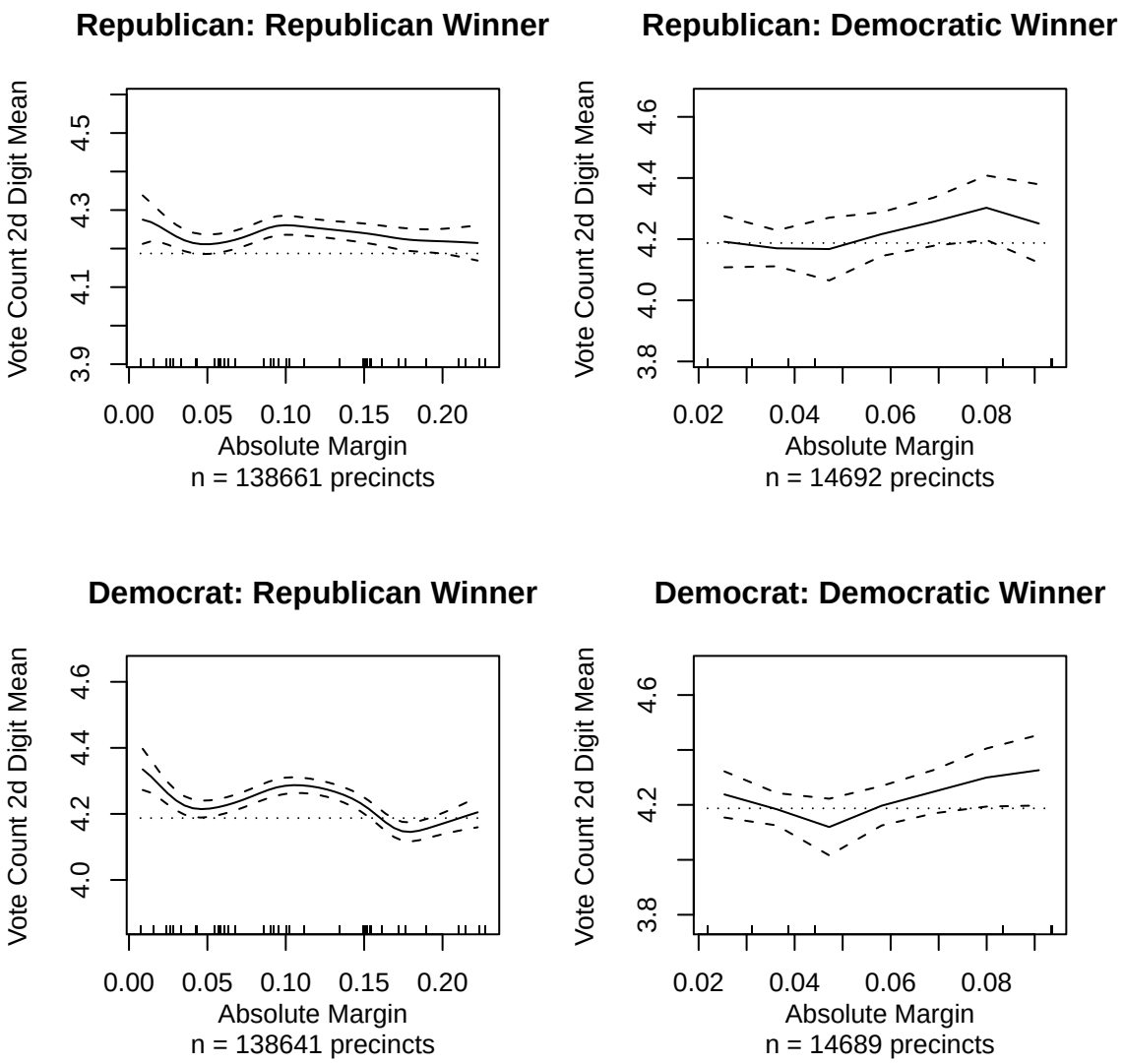

Figure 4: Vote Counts for President, 2000

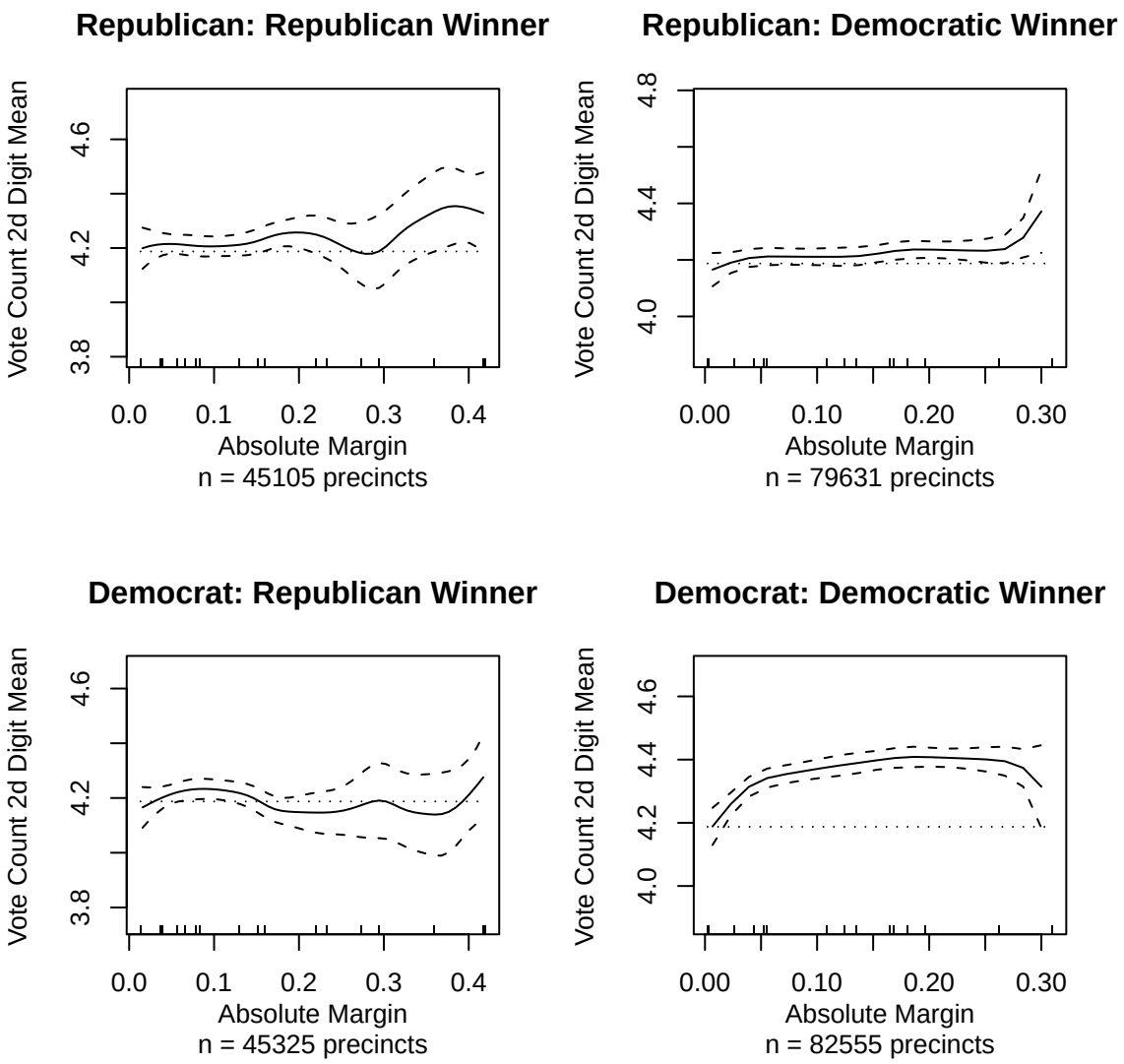

Figure 5: Vote Counts for President, 2004

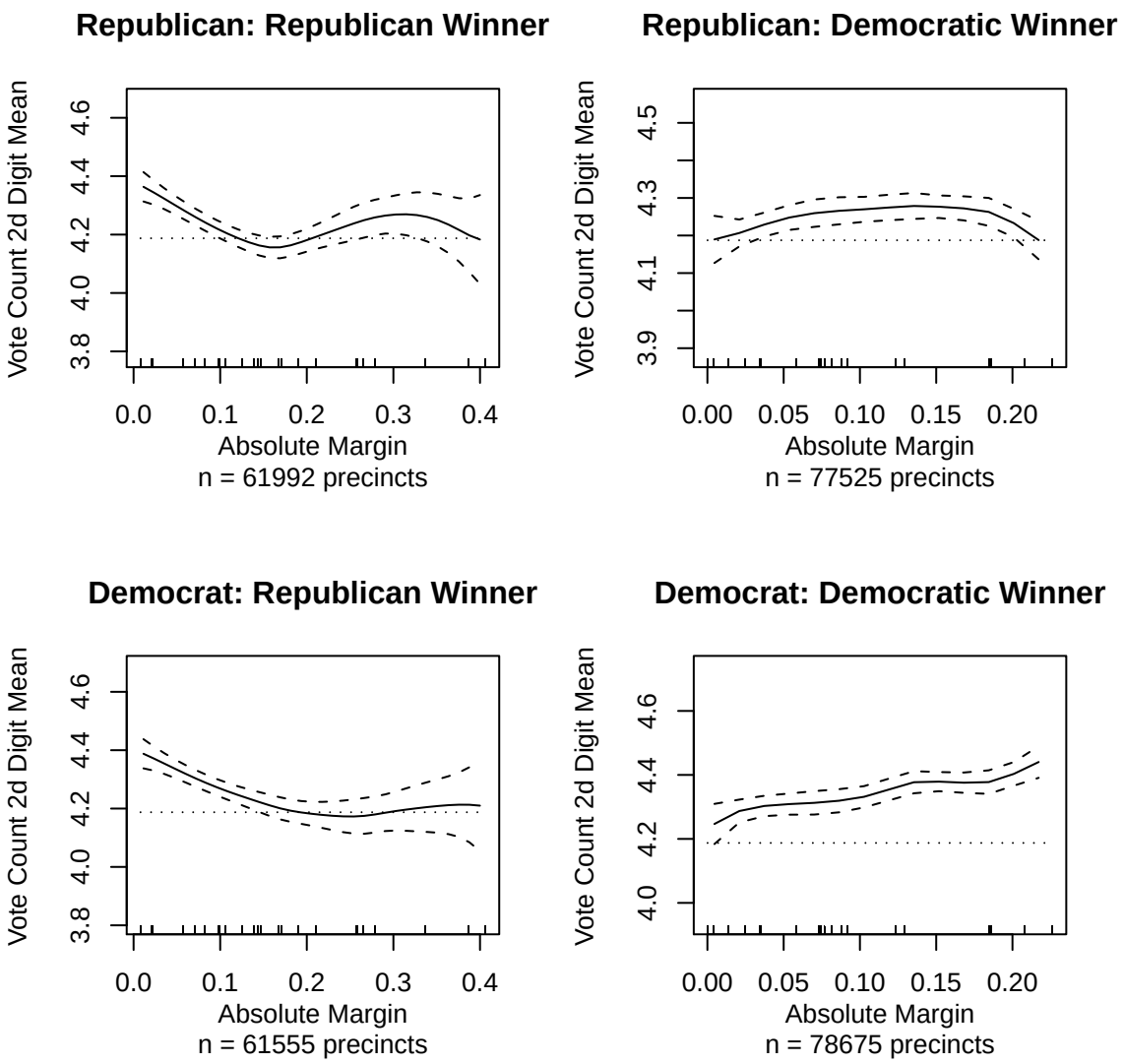

Figure 6: Vote Counts for United States Representative, 1984–90

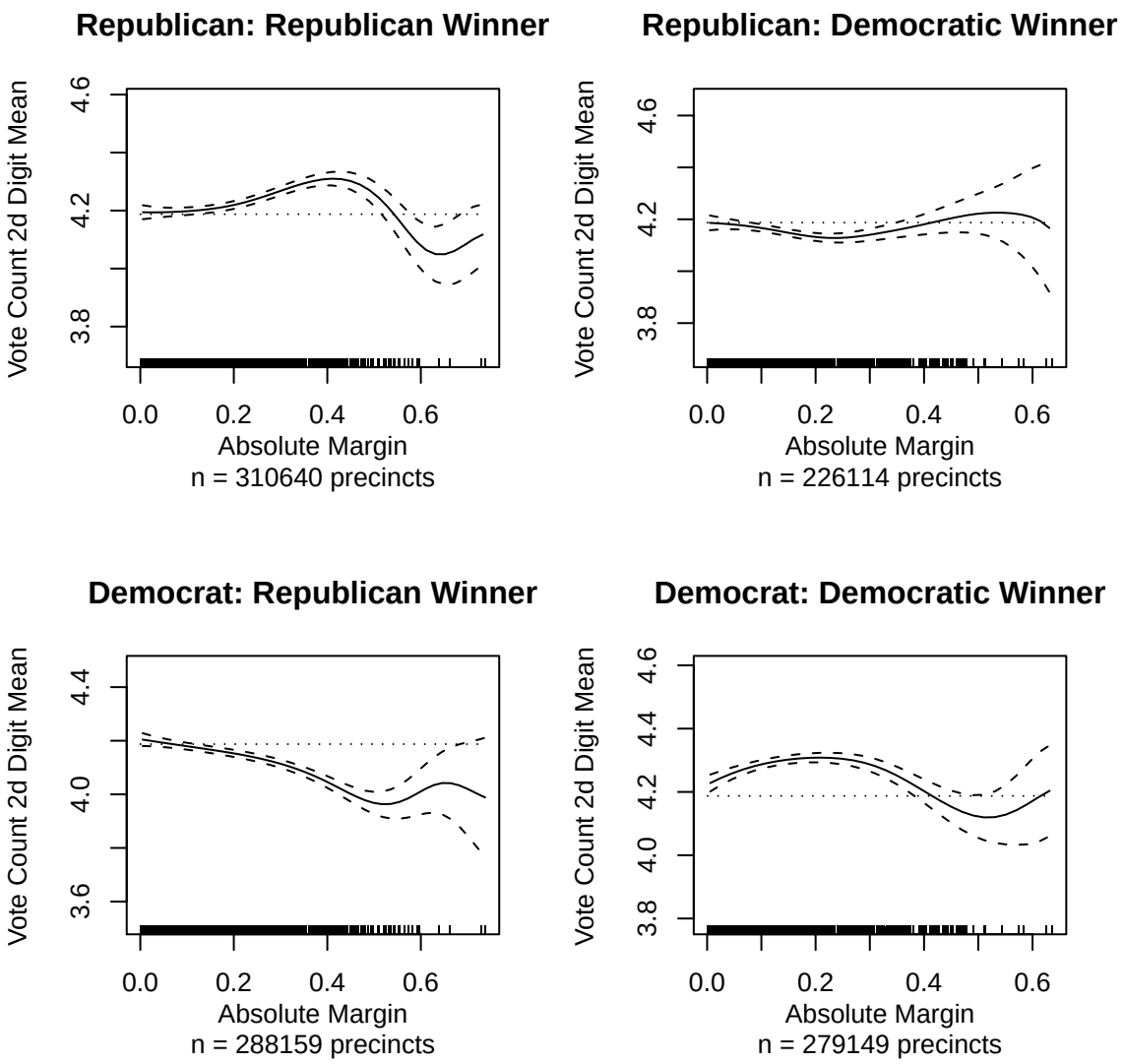

Figure 7: Vote Counts for United States Senate, 1984–90

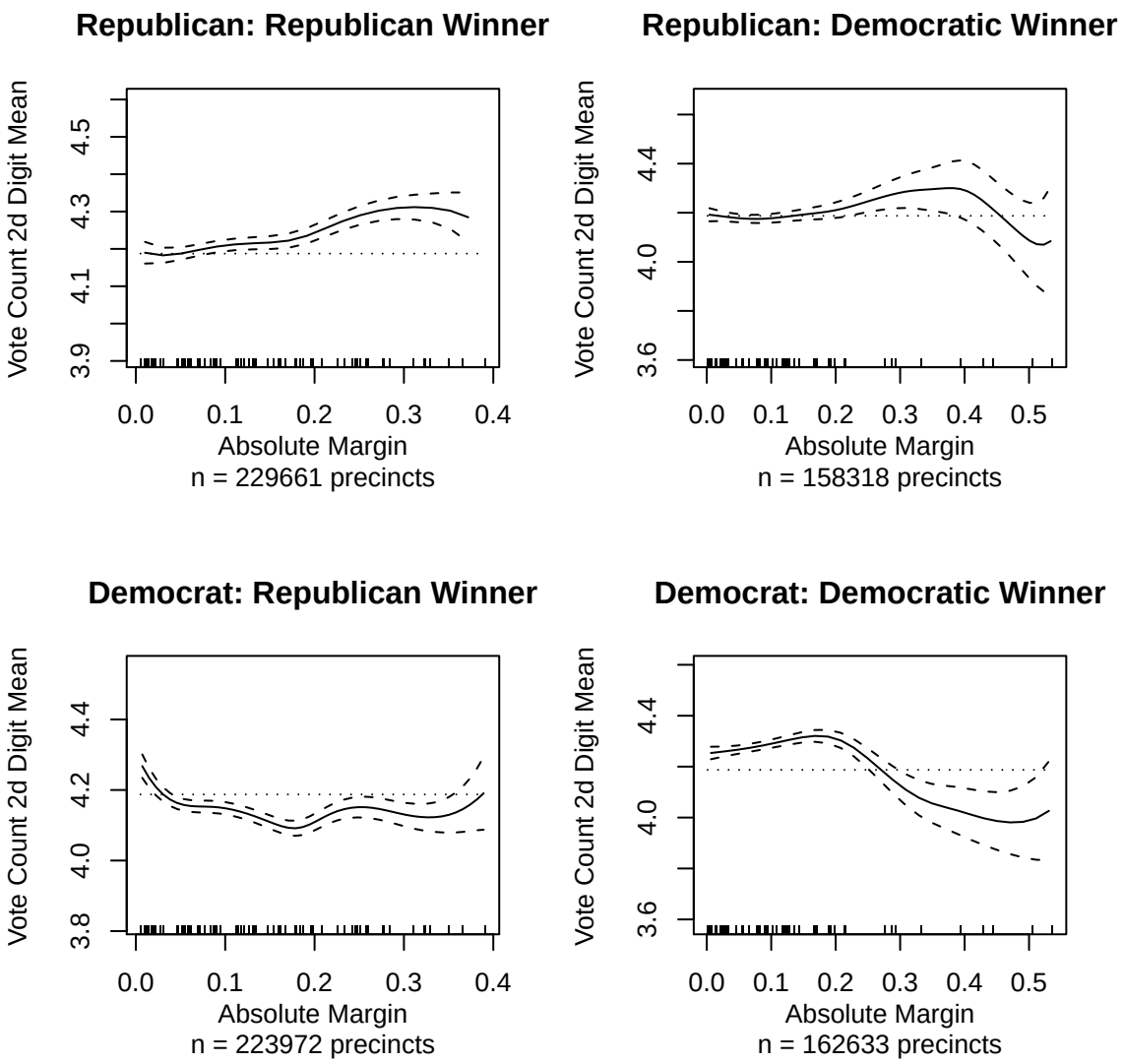

Figure 8: Vote Counts for United States Representative, 2006–08

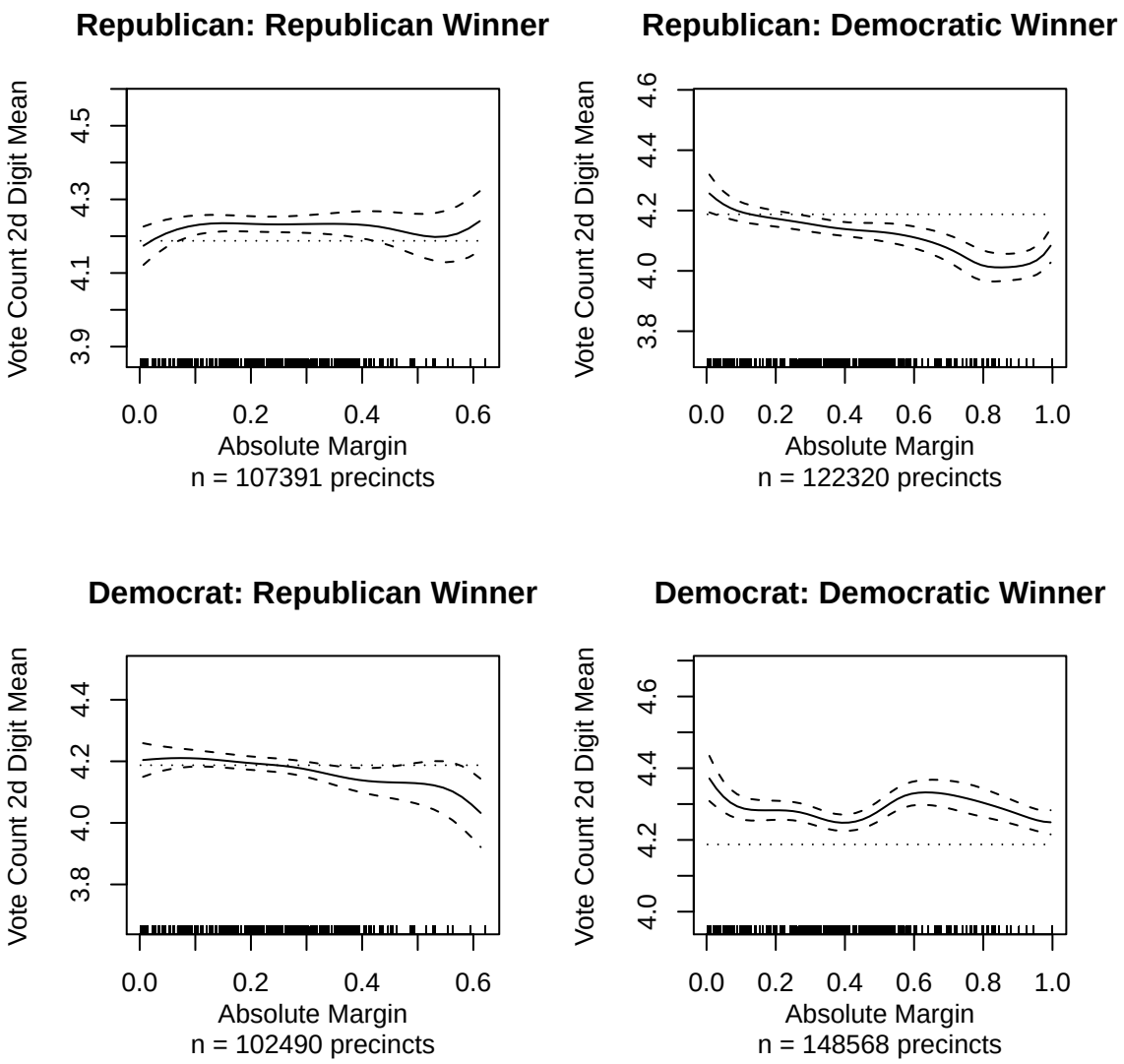

Figure 9: Vote Counts for United States Senate, 2006–08

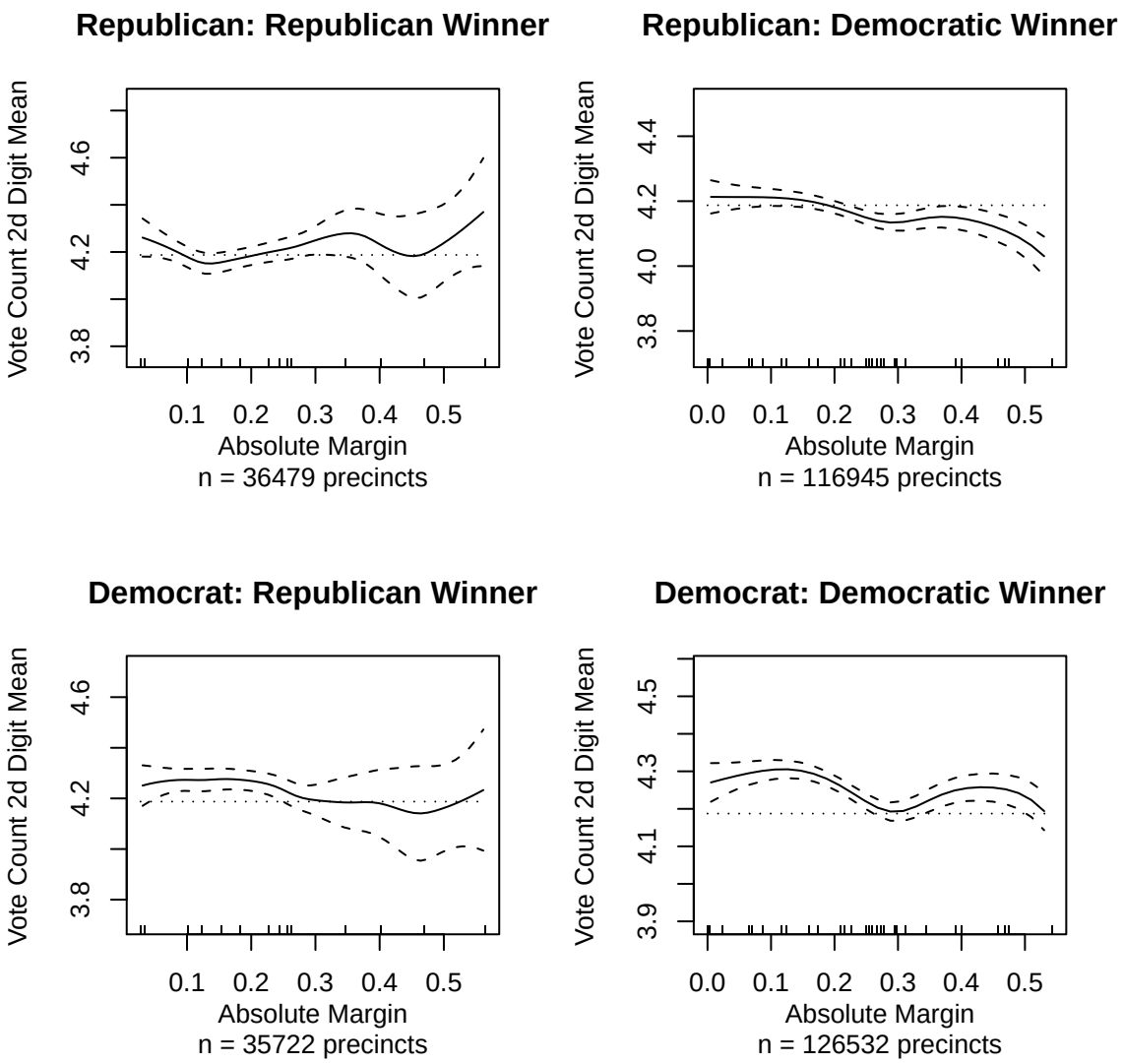

Figure 10: Vote Counts for State House and Senate, 1984–90

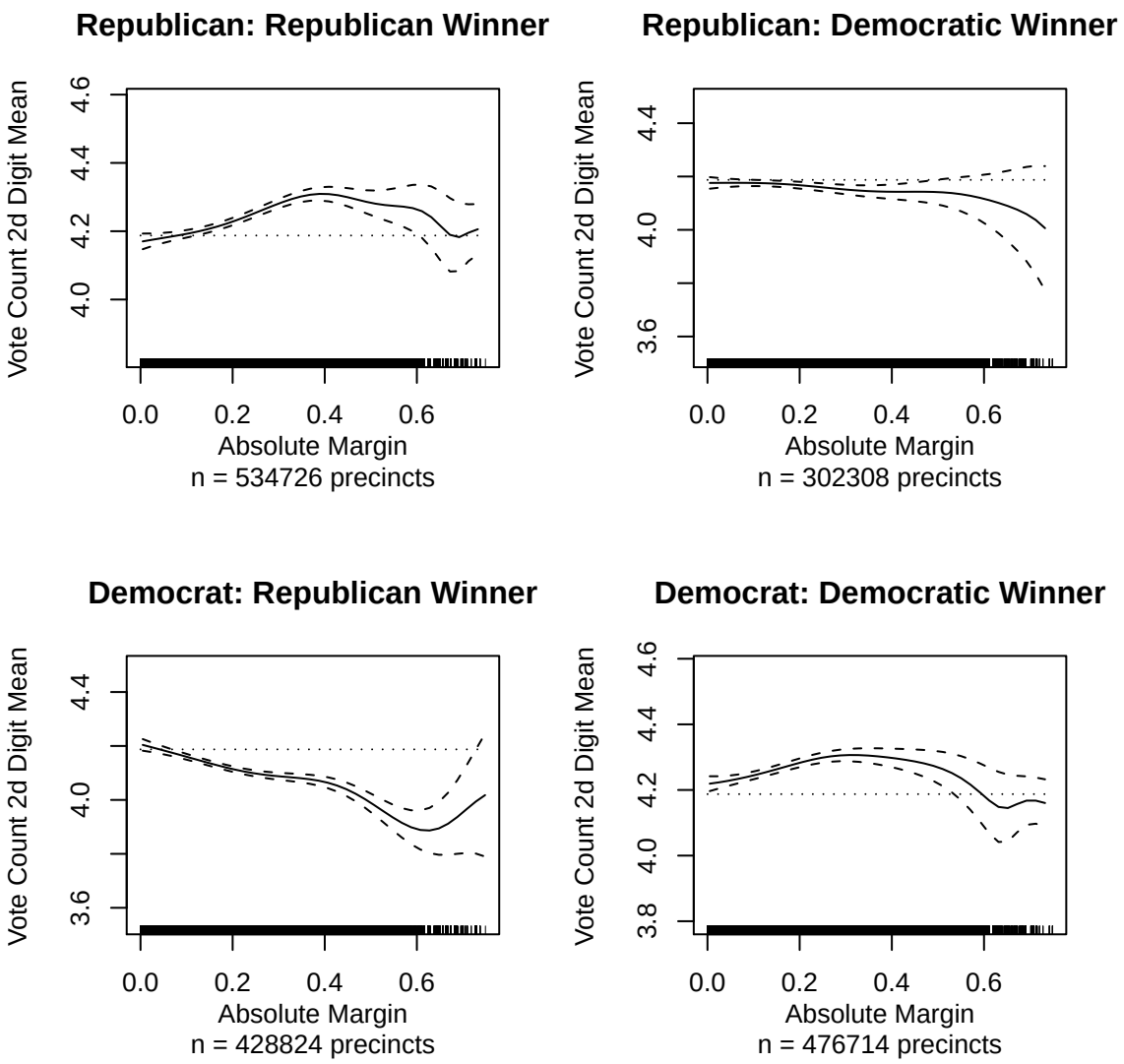

Figure 11: Vote Counts for State House and Senate, 2006–08

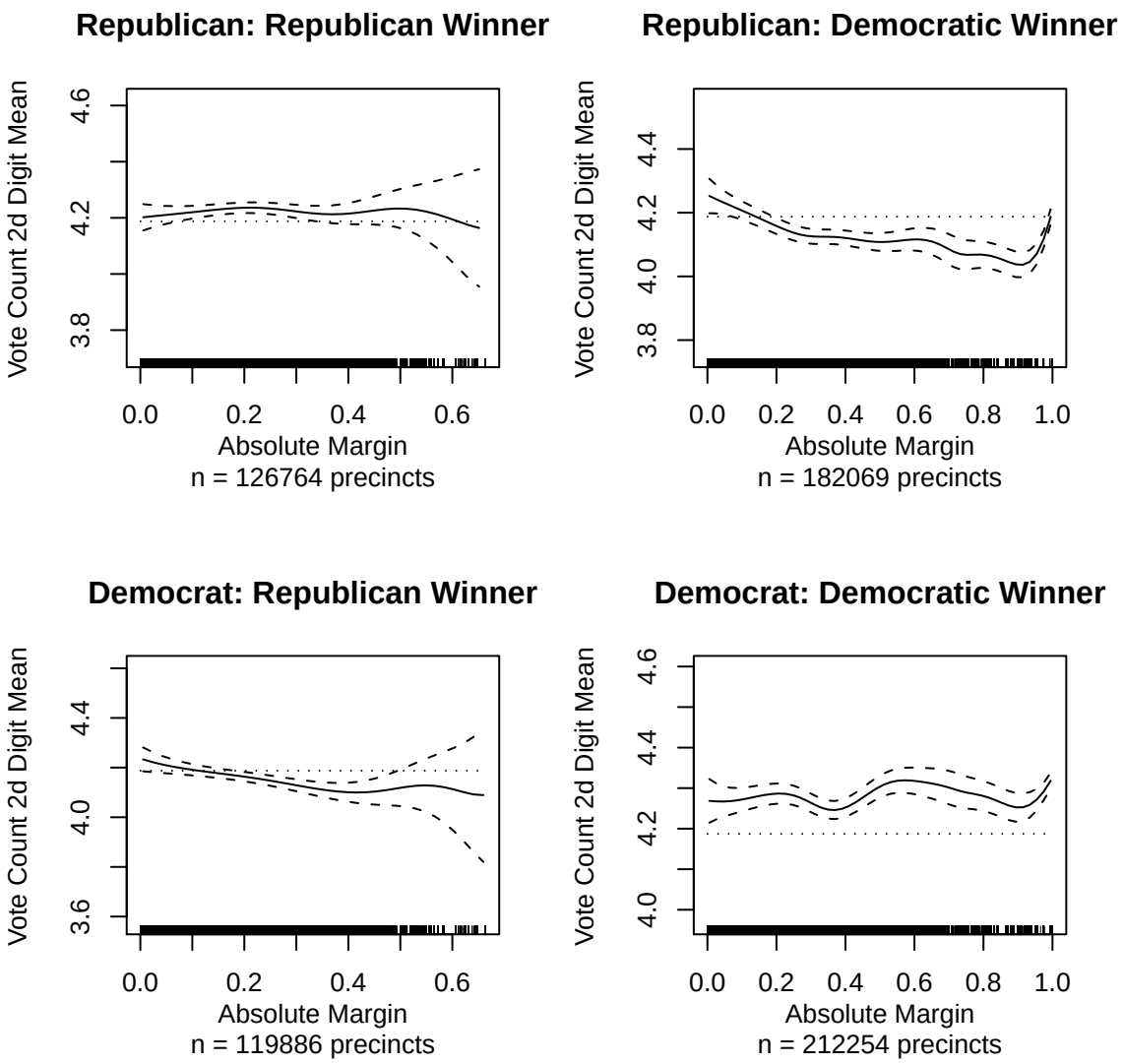

Figure 12: Vote Counts for Governor, 1984–90

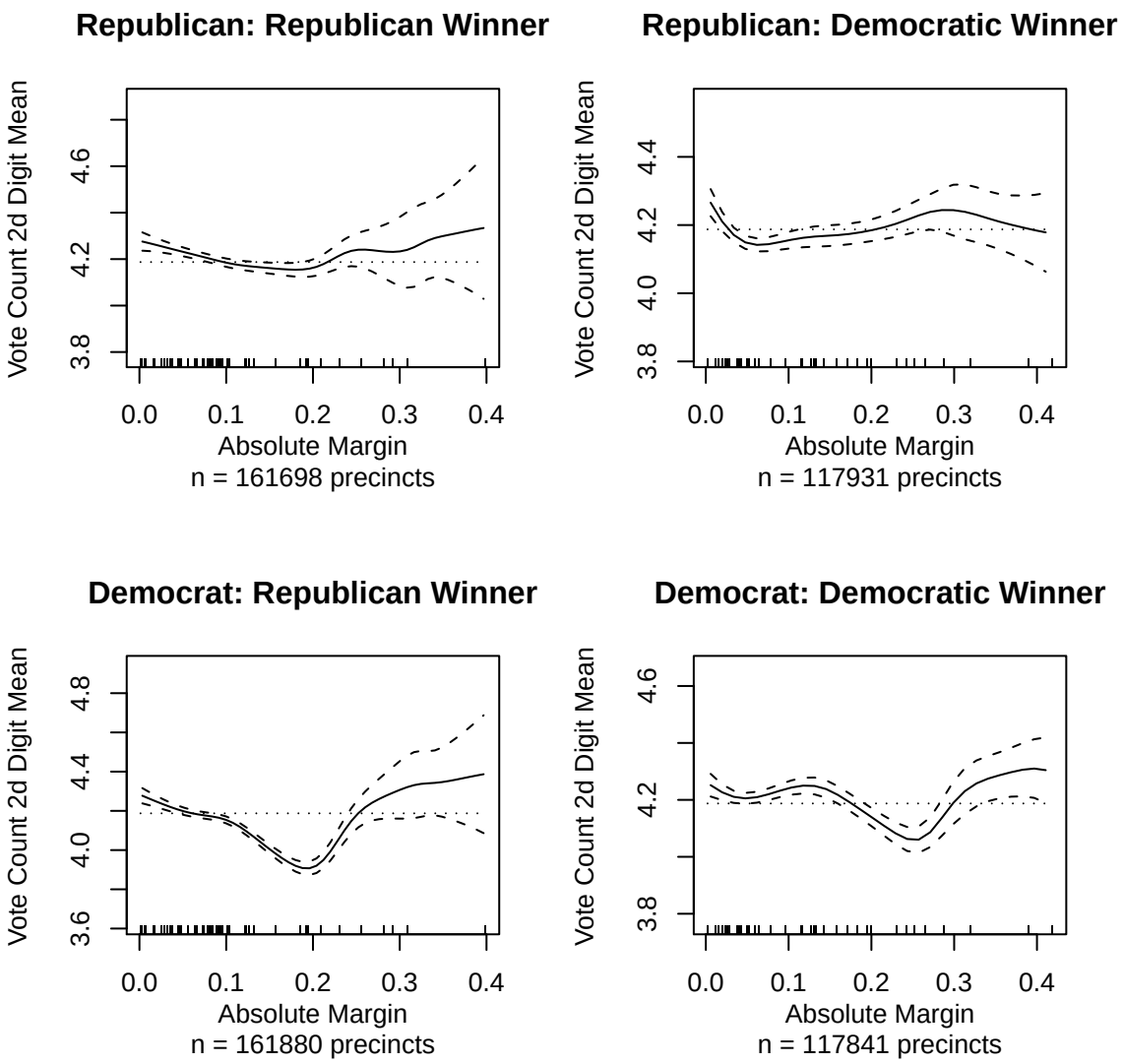

Figure 13: Vote Counts for Governor, 2006–08