# Lost Votes and Posterior Multimodality in the eforensics 

## Model*

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#### Abstract

eforensics produces valid estimates of the number of fraudulent votes at each observed aggregation unit (e.g., precinct): while valid for measuring malevolent distortions of electors' intentions, eforensics estimates are not perfect. The model can produce false positive estimates (or false negative estimates) for eforensics-frauds in two circumstances that often occur: both strategic behavior and lost votes can cause the eforensics model to misestimate malevolent distortions of elector's intentions. Lost votes can occur due to malevolent actions such as voter intimidation but also as a kind of strategic behavior. The eforensics model specification cannot validly accommodate lost votes, so the occurrence of lost votes implies that the model is misspecified. Such misspecification can produce erroneous eforensics-frauds estimates. I present evidence that lost votes can sometimes be detected by assessing multimodality in the posterior distribution of the eforensics model's mixture probability parameters. I present evidence that lost votes can be detected when votes are lost asymmetrically between the leader (the alternative the eforensics model specifies benefits from frauds) and opposition (all other alternatives). I offer two tests that can be routinely used with eforensics when the model parameters are estimated using multiple MCMC chains. Surprisingly small amounts of multimodality can mean that eforensics estimates are being affected by misspecification due to lost votes.


## 1 Introduction

It is difficult to measure election frauds, even more so when all one has to work with are basic summaries of the election such as the number of eligible voters and the number of votes received by ballot alternatives (say candidates) at each aggregation unit. Such measurement is the core mission for election forensics - the field devoted to using statistical methods to determine whether the results of an election accurately reflect the intentions of the electors (cf. Mebane 2008). ${ }^{1}$ By referring to measurement I go beyond trying merely to detect election anomalies (e.g. Myagkov, Ordeshook and Shaikin 2009; Mebane 2014; Montgomery, Olivella, Potter and Crisp 2015; Rozenas 2017; Cantú 2019). Recently methods attempting frauds measurement have been proposed (Klimek, Yegorov, Hanel and Thurner 2012; Klimek, Jiménez, Hidalgo, Hinteregger and Thurner 2018; Zhang, Alvarez and Levin 2019). I use the new statistical model eforensics (Ferrari, Mebane, McAlister and Wu 2019) to argue that the key challenges to frauds measurement are unobservable information and ambiguity-and eforensics goes a long way towards overcoming these. In particular in this paper I examine consequences lost votes have for eforensics estimation, with sidelong attention to contributions made by strategic elector behavior.

What are election frauds? Mindful of both polyarchy (Dahl 1956) and social choice theory (Riker 1982), election forensics refers to election results accurately reflecting the intentions of the electors. Frauds thwart such accurate reflection. I define election frauds as malevolent distortions of electors' intentions that change or potentially can change election outcomes. To distinguish frauds from mere failures of election administration or other accidents, we might require such thwarting to result from undemocratic actions such as are undertaken by authoritarians (e.g. Lehoucq 2003; Schedler 2006; Svolik 2012; Simpser 2013; Norris 2014). Procedural failures might distort intentions, and procedural failures might be planned to distort. But I say the distortions are the frauds, not the procedural failures per se. Many other approaches try to detect and assess what I call "procedural frauds," but

[^0]my aim with eforensics is to measure "realized frauds." How many votes are misdirected or misallocated due to malevolent distortions of electors' intentions? In section 2.2 I will compare eforensics estimates to an example of a consequential process that was based on detecting procedural frauds to support my claim that eforensics estimates are valid (although not perfect).

Reference to electors' intentions points to a key unobservable: we can't observe electors' intentions. In politics people act not merely based on what they prefer but also based on what they expect other people to do: by acting in part based on their expectations regarding others, electors act strategically (e.g. Stephenson, Aldrich and Blais 2018). Strategic behavior must be admitted in any election system that satisfies criteria for democracy such as the Gibbard-Satterthwaite theorem expresses (Riker 1982). Acting strategically does not necessarily mean that electors' actions differ from what their sincere actions would be (Kawai and Watanabe 2013), but they might.

Unobservable intentions imply that frauds measurement faces a fundamental ambiguity. Any measurement effort focuses on empirical patterns. Malevolent distortions of intentions-frauds-produce patterns via a mechanism similar to what happens when there is strategic behavior, that is when there is normal politics (see section 2.3). I show that eforensics provides valid but imperfect measures of frauds, the imperfection largely stemming from such ambiguities. Features of eforensics may support discriminating the frauds.

Manifestations of election frauds are many and varied: beyond tampering with vote tallies there are deploying fake voters and votes, voter intimidation, election violence, voter suppression, misinformation and more (e.g. Birch 2011; Wang 2012; Rundlett and Svolik 2016; Jamieson 2018). Audits can detect some kinds of tampering (e.g. Electoral Complaints Commission 2010; Alvarez, Morrell, Rivest, Stark and Stewart 2019), but with eforensics the aim is to measure distortions including those that might make even procedurally accurate vote tallies not match electors' intentions.

Strategic behavior is no less diverse. Strategies to consider that relate to cases discussed in this paper include wasted-vote strategies (Cox 1994), threshold insurance and two-vote strategies (Shikano, Herrmann and Thurner 2009; Harfst, Blais and Bol 2018), bandwagons (Berch 1989), coordinating split-ticket voting (Alesina and Rosenthal 1995; Mebane 2000), majority-or-runoff strategies (Bouton and Gratton 2015) and strategic abstention (Cox and Munger 1989; Feddersen and Pesendorfer 1999). For the most part in this paper, because the focus is on lost votes, I do not elaborate details describing how the various strategies bear on each election I mention.

If not because of strategic considerations, the most common reasons for lost votes are failures in election administration. Administrative weaknesses include bad ballots and voter identification requirements that produce lost votes. I show that the imperfections eforensics exhibits as a model for measuring frauds need not prevent its being used if it is used with appropriate attention to nuances linked to estimated parameter values.

After motivating and describing the Bayesian formulation of eforensics and its Markov Chain Monte Carlo (MCMC) estimation approach, I present one case that supports believing that eforensics supplies valid measures of malevolent distortions of electors' intentions. Then I use a notional individual-level formulation to discuss ambiguities relating to the eforensics model. Then the discussion turns to several cases chosen to bring out aspects of what eforensics does in the presence of lost votes. Considerations about strategic elector behavior also inform the discussion.

## 2 eforensics Model Motivation

Statistical approaches based only on counts of electors and votes are challenged because neither electors' preferences, strategies nor information are observed-nor is whether anyone's actions are malevolent-yet the election forensics task is to assess whether electors' intentions are accurately reflected in the election outcome. eforensics is based
on an explict model: functional form commitments stand in place of features it is impossible to observe. What's the problem?

The eforensics model assumes that if there are no frauds then each elector decides whether to vote and, if so, for whom in a way that can be represented by two binary choices governed by Bernoulli probabilities. The turnout choice is between "vote" and "abstain," and the vote choice decision is between "leader" and "opposition." Conditioning on the number of electors $\left(N_{i}\right)$ at aggregation unit $i$, the number of votes cast is then an overdispersed binomial random variable: the turnout probability averages the electors' probabilities at $i$ with extra variation due to variation across individuals. Conditioning on $N_{i}$, the number of people voting for the leader is an overdispersed binomial with probability being the product of turnout and vote choice probabilities, with extra variation due to variation across individuals.

A key aspect of the mechanism by which frauds distort intentions is that they induce dependencies among individuals' observed votes (see section 2.3 for further discussion). Imagine, for example, that fake votes are added all for the vote leader. The fake votes are dependent. Notionally similar dependence arises if many are coerced to vote for the leader, etc. The eforensics model measures the dependencies using a finite mixture model. One component corresponds to no fraud, one to "incremental fraud" and one to "extreme fraud." The idea for such a mixture structure comes from Klimek et al. (2012), who emphasize how frauds induce multimodal data distibutions, ${ }^{2}$ which in their view is the manifestation of the dependencies we can observe given aggregate count data.

The problem is that other factors in politics induce dependence, and discriminating dependence that traces to frauds from dependence that originates elsewhere is challenging. Strategic behavior generically presents the most difficult problem because both strategic behavior and frauds induce dependence. For instance, in a Nash equilibrium each elector considers and responds to all others' expected votes. Generally strategic behavior implies

[^1]there are dependencies among all strategic electors' behavior, with some electors acting systematically similarly and others acting systematically in opposition. Both frauds and strategic behavior can involve votes being changed-with frauds it's some malefactor that changes votes while with strategic behavior individual electors may change their own votes from what each would do if acting sincerely. How to discriminate effects of strategic behavior from effects of frauds is a question.

Features of election administration can also induce dependence among votes but not originate with malevolent actions. E.g., variation in ballot quality or voting equipment provision can induce widespread confusion or delays that lead to voting errors or decreased turnout (e.g. Mebane 2004; Pettigrew 2017). Multiple sources of dependence mean eforensics estimates may be ambiguous as far as interpretations in terms of frauds-malevolent distortions - are concerned.

### 2.1 Model Specification

In eforensics electors either vote or abstain, and vote choices are reduced to two options: one candidate or other ballot alternative is the "leader"; the remaining alternatives are grouped as "opposition." Frauds benefit the leader. Some votes are transferred to the leader from opposition ("stolen"), and some are taken from nonvoters ("manufactured").
eforensics model observed variables Observed data for $n$ aggregation unit observations indexed by $i=1, \ldots, n$ include the total number of vote-eligible persons $\left(N_{i}\right)$, the number of votes for the leader $\left(W_{i}\right)$ and the number of votes cast $\left(V_{i}\right)$. The model conditions on $N_{i}$. The number of abstentions is $A_{i}=N_{i}-V_{i}$. The number of votes for opposition plays no direct role in the model but to clarify definitions I specify that as $O_{i}=V_{i}-W_{i}$. Covariates $\left\{x_{i}^{\nu}, x_{i}^{\tau}, x_{i}^{\nu}, x_{i}^{v}\right\}_{i=1}^{n}$, described below, are also observed.
eforensics model The incidence of eforensics-frauds is indicated by unobserved variables $Z_{i} \in\{1,2,3\}$, where $\operatorname{Prob}\left(Z_{i}=k\right)=\pi_{k}, k \in\{1,2,3\}: Z_{i}=1$ means "no fraud",
$Z_{i}=2$ means "incremental fraud," and $Z_{i}=3$ means "extreme fraud." The prior for $\pi_{k}$ ensures that $\pi_{1}$ is weakly largest:

$$
\begin{align*}
& \tilde{\pi}_{1} \sim U(0,1) ; \quad \tilde{\pi}_{2} \sim U\left(0, \tilde{\pi}_{1}\right) ; \quad \tilde{\pi}_{3} \sim U\left(0, \tilde{\pi}_{1}\right)  \tag{1a}\\
& \pi_{j}=\frac{\tilde{\pi}_{j}}{\tilde{\pi}_{1}+\tilde{\pi}_{2}+\tilde{\pi}_{3}}, \quad j \in\{1,2,3\} \tag{1b}
\end{align*}
$$

I use this prior to deter label switching (Grün and Leisch 2009).
With $\tau_{i}$ being the unobserved true proportion of electors who vote and $\nu_{i}$ being the unobserved true proportion of votes cast that are for the leader, the magnitudes of eforensics-frauds are determined using proportions

$$
\begin{align*}
& p_{t i}= \begin{cases}0, & \text { if } Z_{i}=1 \\
\iota_{i}^{M}\left(1-\tau_{i}\right), & \text { if } Z_{i}=2, \\
v_{i}^{M}\left(1-\tau_{i}\right), & \text { if } Z_{i}=3\end{cases}  \tag{2a}\\
& p_{w i}= \begin{cases}0, & \text { if } Z_{i}=1 \\
\iota_{i}^{M}\left(1-\tau_{i}\right)+\iota_{i}^{S} \tau_{i}\left(1-\nu_{i}\right), & \text { if } Z_{i}=2 \\
v_{i}^{M}\left(1-\tau_{i}\right)+v_{i}^{S} \tau_{i}\left(1-\nu_{i}\right), & \text { if } Z_{i}=3\end{cases} \tag{2b}
\end{align*}
$$

where $p_{t i}$ is fraudulent turnout, and $p_{w i}$ is fraudulent vote choice. $\iota_{i}^{M}$ or $v_{i}^{M}$ are the proportions of true abstentions that are instead counted as votes cast (manufactured votes), and $\iota_{i}^{S}$ or $v_{i}^{S}$ are the proportions of true votes for opposition that are instead counted as votes for the leader (stolen votes). So the proportion of electors observed abstaining expressed in terms of eforensics-frauds is $a_{i}^{*}=1-\tau_{i}-p_{t i}$, and the proportion voting for the leader is $w_{i}^{*}=\nu_{i} \tau_{i}+p_{w i}$. The likelihood for observations $\left\{A_{i}, W_{i}\right\}_{i=1}^{n}$, which conditions on $\left\{N_{i}, x_{i}^{\nu}, x_{i}^{\tau}, x_{i}^{L}, x_{i}^{\nu}\right\}_{i=1}^{n}$, is a product of binomial distributions each having $N_{i}$
trials and binomial probability respectively $a_{i}^{*}$ and $w_{i}^{*}$ :

$$
\begin{equation*}
\mathcal{L}=\prod_{i=1}^{n}\binom{N_{i}}{A_{i}}\left(a_{i}^{*}\right)^{A_{i}}\left(1-a_{i}^{*}\right)^{N_{i}-A_{i}}\binom{N_{i}}{W_{i}}\left(w_{i}^{*}\right)^{W_{i}}\left(1-w_{i}^{*}\right)^{N_{i}-W_{i}} . \tag{3}
\end{equation*}
$$

The unobserved proportions $\tau_{i}, \nu_{i}, \iota_{i}^{M}, \iota_{i}^{S}, v_{i}^{M}$ and $v_{i}^{S}$ are defined using observed covariate vectors $x_{i}^{\tau}, x_{i}^{\nu}, x_{i}^{\iota}, x_{i}^{v}$, coefficient vectors $\gamma, \beta, \rho_{M}, \rho_{S}, \delta_{M}, \delta_{S}$ and random effects $\kappa_{i}^{\nu}, \kappa_{i}^{\tau}, \kappa_{i}^{\iota M}, \kappa_{i}^{\iota S}, \kappa_{i}^{\nu M}, \kappa_{i}^{\nu S}:$

$$
\begin{align*}
\tau_{i} & =\frac{1}{1+\exp \left[-\left(\gamma^{\top} x_{i}^{\tau}+\kappa_{i}^{\tau}\right)\right]}  \tag{4a}\\
\nu_{i} & =\frac{1}{1+\exp \left[-\left(\beta^{\top} x_{i}^{\nu}+\kappa_{i}^{\nu}\right)\right]}  \tag{4b}\\
\iota_{i}^{l} & =\frac{k}{1+\exp \left[-\left(\rho_{l}^{\top} x_{i}^{l}+\kappa_{i}^{\iota l}\right)\right]}, l \in\{M, S\}  \tag{4c}\\
v_{i}^{l} & =+\frac{1-k}{1+\exp \left[-\left(\delta_{l}^{\top} x_{i}^{v}+\kappa_{i}^{\nu l}\right)\right]}, l \in\{M, S\} \tag{4d}
\end{align*}
$$

where $k=.7$. Each coefficient in $\gamma, \beta, \rho_{M}, \rho_{S}, \delta_{M}, \delta_{S}$ has an independent Normal prior $(N(0,1 / 10000))$. Each $\kappa_{i}^{\xi}, \xi \in\{\nu, \tau\}$, is an unobserved variable that for unknown mean $\mu^{\kappa \xi}$ and standard deviation $\sigma^{\kappa \xi}$ is assumed to have as prior the Normal distribution $\kappa_{i}^{\xi} \sim N\left(\mu^{\kappa \xi}, \sigma^{\kappa \xi}\right)$ with $\mu^{\kappa \xi} \sim N(0,1), \sigma^{\kappa \xi} \sim \operatorname{Exp}(5),{ }^{3}$ and likewise for $\kappa_{i}^{\iota M}, \kappa_{i}^{\iota S}, \kappa_{i}^{v M}$ and $\kappa_{i}^{v S}$. In $\tau_{i}$ and $\nu_{i}$ random effects $\kappa_{i}^{\tau}$ and $\kappa_{i}^{\nu}$ capture overdispersion, and in $\iota_{i}^{M}, \iota_{i}^{S}, v_{i}^{M}$ and $v_{i}^{S}$ random effects $\kappa_{i}^{\iota M}, \kappa_{i}^{\iota S}, \kappa_{i}^{v M}$ and $\kappa_{i}^{v S}$ capture extra variation in observation-level frauds.
estimation eforensics implements MCMC using Metropolis-Hastings (Plummer, Stukalov and Denwood 2016) with four chains (Denwood 2016), using Markov Chain Monte Carlo Standard Error (MCMCSE) (Flegal, Haran and Jones 2008; Flegal and Hughes 2012; Gong and Flegal 2016) for a stopping rule. Chains run until $\sigma_{\theta_{j}}<.05$ where $\sigma_{\theta_{j}}^{2}$ is the estimated asymptotic variance of the $j^{\text {th }}$ component of parameter vector $\boldsymbol{\theta}$, computing $\sigma_{\theta_{j}}^{2}$ using consistent nonoverlapping batch means (Jones, Haran, Caffo and Neath 2006).

[^2]Observation $i$ is classified as type $\tilde{Z}_{i} \in\{1,2,3\}$ if a plurality of MCMC iterations have $Z_{i}=\tilde{Z}_{i}$. Using indicator function $\mathcal{I}(\cdot)$ the number of eforensics-fraudulent observations is $H=\sum_{i=1}^{n} \mathcal{I}\left(Z_{i} \in\{2,3\}\right)$, and the proportion is $\varphi=H / n$. Numbers of eforensics-fraudulent voters and votes for the leader at $i$ are $F_{t i}=p_{t i} N_{i}$ and $F_{w i}=p_{w i} N_{i}$, with totals $F_{t}=\sum_{i=1}^{n} F_{t i}$ and $F_{w}=\sum_{i=1}^{n} F_{w i}$.

### 2.2 Illustration of eforensics Validity

We present one example that illustrates that eforensics estimates are valid measures of malevolent distortions of electors' intentions. Starting in the next section I'll emphasize imperfections, including especially imperfections that relate to lost votes.

The example is the 2017 National Assembly election in France (Kuhn 2018). The Conseil Constitutionnel issued 505 decisions concerning 307 districts, including eight decisions to annul a district's election. Seven of those decisions addressed elections that were decided by second-round elections, while one addressed a first-round decision. We show the second-round decisions relate positively to eforensics-fraudulent votes: having more eforensics-fraudulent votes in a legislative district, relative to the margin of victory in the district, is associated with a higher probability of the district's election being annulled. eforensics estimation for second-round votes uses bureaux de vote aggregation units, ${ }^{4}$ and $W_{i}$ contains district winner votes. The eforensics specification pools all districts: the model includes all bureaux with district fixed effects specified for turnout and vote choice (in $x^{\tau}$ and $x^{\nu}$ of (4)(a) and 4)(b)).

Figure 1 shows a scatterplot of the round 2 election results in the form of turnout and leader proportions $\left(t_{i}\right.$ and $\left.w_{i}\right)$, with histograms along the margins and a two-dimensional empirical density shown behind the scatterplot's points. The plot of the original data in Figure 1(a) has the striking feature that most of the bureaux have more than half the votes counted for the leader. This pattern is partly explained because in the second round most

[^3]districts have only two candidates. Figure 1(b) shows the data after each bureau has its district's mean removed-i.e., district fixed effects are removed. These residualized observations are extremely skewed but less so than are the original proportions.

According to the eforensics model mixture probabilities, eforensics-frauds are rare (estimates in Table 1): $\pi_{1}$ has posterior mean .990 with HPD $95 \%$ interval [.987, .992]; $\pi_{2}=.0102[.00765, .0134] ; \pi_{3}=.000106$ [.0000301, .000191$]$. Among frauds magnitude parameters $\rho_{M 0}<0$ and $\rho_{S 0}<0$ while $\delta_{M 0}$ and $\delta_{S 0}$ do not differ from the prior means of zero. In line with the mixture probabilities, $\varphi=484 / 68760=.00704$ is small. Summed over all districts' posterior means $F_{t}=22471.8$ and $F_{w}=28719.6$ are small as a proportion of leaders' votes $\sum_{i=1}^{n} W_{i}=10970881: 28719.6 / 10970881=.00262$.

We use Poisson and binomial logistic regressions to assess eforensics-frauds estimates' relation to Conseil actions (Table 2). One model's outcome is the number of cases in each district (Table 2(A)). The other's is whether annulment occurs given that there is a case (Table 2(B)). ${ }^{5}$ Regressors are district sums of manufactured $\left(F_{t}\right)$, stolen $\left(F_{w}-F_{t}\right)$ or total eforensics-fraudulent votes in each district, each divided by the margin $M$ between votes for the first- and second-place candidates in each district. Language in the annulment decisions guides how we use $M$ to normalize eforensics-fraudulent votes. Frequently decisions say, "in view of the small difference in votes between the two candidates present in the second ballot, it is necessary, without there being any need to examine the other complaints, to annul the contested electoral operations." ${ }^{6} M$ being interesting to the court (Klaver and Mebane 2022), we use regressors $F_{t} /(M+1),\left(F_{w}-F_{t}\right) /(M+1)$ and $F_{w} /(M+1)$.

As the note at the end of Table 2 reports, $F_{t} /(M+1)$ and $\left(F_{w}-F_{t}\right) /(M+1)$ are strongly correlated, so in each regression specification we include only one regressor at a time. ${ }^{7}$ While $F_{t} /(M+1)$ and $\left(F_{w}-F_{t}\right) /(M+1)$ are each strongly positively associated

[^4]with the number of cases pertaining to each district, AIC suggests it is sufficient and best to represent case occurrence as depending solely on $M$ : an increasing district margin decreases the probability that there is a case. ${ }^{8}$

Coefficient estimates in Table 2(B) show that the probability of annulments increases with each of the aspects of eforensics-frauds used as regressor. If the estimated coefficients are multiplied by the respective upper bounds of the ranges for $F_{t} /(M+1)$, $\left(F_{w}-F_{t}\right) /(M+1)$ and $F_{w} /(M+1)$ shown in Table 2's note, then the posterior means $.795(6.12)=4.9,3.73(1.30)=4.9$ and $3.73(7.775)=29.0$ reveal hefty spans for the implied probabilities. AIC evaluated at the posterior mean is smallest for $\left(F_{w}-F_{t}\right) /(M+1)$ in Table 2(B)(f), but the differences from the specifications that include instead $F_{t} /(M+1)$ or $F_{w} /(M+1)$ are too small to emphasize all that much. A detail not shown in Table 2 is that of the ten largest normalized eforensics-fraudulent vote totals six occur for districts that had elections annulled.

Estimated eforensics frauds closely and strongly correspond to National Assembly annulment decisions by the Conseil Constitutionnel in 2017. As Klaver and Mebane (2022) reports similar findings occur for 2012 National Assembly elections.

## 2.3 eforensics Model Motivation and Core Ambiguity

We use notional individual-level functional forms to illuminate how the eforensics model works to estimate malevolent distortions of elector intentions and to describe how and why the model specification is subject to several kinds of ambiguities and misestimates. Strategic elector behavior and lost votes are the main challenges discussed.

Write the individual-level vote choice (or turnout) probability for elector $j$ in

[^5]aggregation unit (say precinct) $i$ in general form as
\[

$$
\begin{equation*}
\zeta_{i j}=\frac{1}{1+\exp \left[-\left(b_{i j}^{\top} x_{i j}\right)\right]} \tag{5}
\end{equation*}
$$

\]

for ideally observed covariate vector $x_{i j}$ and coefficient vector $b_{i j}$. Imagine that the specification of $x_{i j}$ is such that the probabilities $\zeta_{i j}$ are independent across electors. This might be thought of as the simplest kind of individual-level reality if there are not malevolent distortions of elector intentions. Contrast this with an alternative general individual-level specification with malevolent distortion components $D_{i}$ :

$$
\begin{equation*}
\zeta_{i j}^{*}=\frac{1}{1+\exp \left[-\left(b_{i j}^{\top} x_{i j}+D_{i}\right)\right]} . \tag{6}
\end{equation*}
$$

If the vote probabilities under consideration are probabilities of voting for the leader, then precincts that have adding frauds (as in (2a) and (2b) given the definitions of $a_{i}^{*}$ and $w_{i}^{*}$ ) will have $D_{i}>0$. This means that for pairs of precincts $i$ and $i^{\prime}$ that have frauds, with $i \neq i^{\prime}, \operatorname{cov}\left(D_{i}, D_{i^{\prime}}\right) \neq 0$. Not only are the terms $D_{i}$ correlated across precincts that have malevolent distortions, but all electors in each such precinct $i$ are perturbed by the same impulse $D_{i} .{ }^{9}$ Hence for precincts $i$ and $i^{\prime}$ that have malevolent distortions, for distinct electors $j \neq j^{\prime}, \operatorname{cov}\left(\zeta_{i j}^{*}, \zeta_{i^{\prime} j^{\prime}}^{*}\right) \neq 0$. This specification represents the way a simple kind of frauds induces dependence among the notionally observed behavior of individual electors.

In the eforensics model specification, the idea is that $\nu_{i}$ (or $\tau_{i}$ ) is approximately $N_{i}^{-1} \sum_{j \in i} \zeta_{i j}$, while the additions in (2a), (2b), $a_{i}^{*}$ and $w_{i}^{*}$ for cases $Z_{i}=2$ and $Z_{i}=3$ via $\iota_{i}^{M}, \iota_{i}^{S}, v_{i}^{M}$ and $v_{i}^{S}$ give approximations to $N_{i}^{-1} \sum_{j \in i} \zeta_{i j}^{*}$. This conveys the sense in which the dependencies among individual electors induced by malevolent distortions-by frauds-identify the mixture probabilities and frauds magnitudes parameters of the

[^6]eforensics model.
Ambiguities arise because other things besides malevolent distortions induce dependence among electors. Three important not necessarily (or necessarily not) fraudulent factors that can induce dependencies are omitted variables, strategic elector behavior and election administration failures.

Omitted Variables Omitted variables generally are less of a problem than one might initially suspect because of the random effects in (4a) and (4b). If the observed covariates $x_{i}^{\tau}$ or $x_{i}^{\nu}$ fail adequately to approximate the ideal covariates $x_{i j}$ as those are aggregated in $N_{i}^{-1} \sum_{j \in i} \zeta_{i j}$ e.g., if $x_{i}^{\tau}$ and $x_{i}^{\nu}$ consist only of the intercept term-then the random effects $\kappa_{i}^{\nu}$ and $\kappa_{i}^{\tau}$ can improve the approximation of $N_{i}^{-1} \sum_{j \in i} \zeta_{i j}$ by (4a) or (4b). With Metropolis-Hastings algorithm MCMC updates, the random effects' Normal priors produce very flexible even if not perfectly flexible posterior distributions. ${ }^{10}$

Strategic Elector Behavior Strategic behavior is not any kind of malevolent distortion of elector intentions; indeed strategic considerations in general are essential parts of electors' intentions (e.g. Riker 1982). But strategic behavior has a feature that makes it potentially a greater challenge for eforensics than are omitted variables. With strategic behavior but no malevolent distortions we might imagine we have idealized individual-level specifications like

$$
\begin{equation*}
\zeta_{i j}^{+}=\frac{1}{1+\exp \left[-\left(b_{i j}^{\top} x_{i j}+S_{i j}\right)\right]}, \tag{7}
\end{equation*}
$$

where $S_{i j}$ represents a strategic contribution based on the expectations elector $j$ has about other electors. In the case of something like Nash equilibrium, $S_{i j}$ will draw on expectations about all other electors. If expectations are at least approximately rational and have such an extensive span, then they induce correlations among electors both within

[^7]and, more directly important for eforensics, between precincts.
These correlations might differ from the correlations that are induced by malevolent distortions. First, for the simplest kinds of strategic behavior like wasted-vote strategies typically all precincts will be involved, not merely some precincts as occurs for the malevolent distortions in scope for eforensics. Second, not all precinct aggregations of the strategic contributions $S_{i j}$ will be positive: if some precincts have most electors who support the leader while other precincts have most electors who support another candidate, they may feature mean values $S_{i}$ that have opposite signs. Nonetheless aggregation of strategic contributions may trigger false positive eforensics estimates. The random effects $\kappa_{i}^{\nu}$ and $\kappa_{i}^{\tau}$ may help filter out the strategic contributions, particularly if the contributions have impacts on almost all precincts. But the independence of the priors used for those random effects may limit the extent to which the random effects can capture the associations across precincts that the aggregated strategic contributions induce.

If several of the averages $S_{i}$ of strategic contributions $S_{i j}$ are negative, then-more than false positives-eforensics estimates may be aliased and exhibit posterior multimodality in ways similar to what can occur with lost votes (as is discussed next), because the eforensics model specification cannot faithfully represent such negative impulses.

Lost Votes The concerning election administration failures are those official actions that impede voting or induce errors in voting or vote tabulation. For example, consider resource allocations that cause excessively long wait times, ballot designs that confuse voters or machine defects that produce mistaken vote counts. We focus on the ways such problems induce lost votes: votes that should have been counted but weren't, perhaps because they weren't cast in the first place (e.g., by electors who can't wait in a long line). With lost votes but no malevolent distortions we might have idealized individual-level specifications
like

$$
\begin{equation*}
\zeta_{i j}^{\#}=\frac{1}{1+\exp \left[-\left(b_{i j}^{\top} x_{i j}+L_{i j}\right)\right]}, \tag{8}
\end{equation*}
$$

where $L_{i j} \leq 0$ is the lost votes component. If a vote is completely lost then $L_{i j}=-\infty$ (or at least $L_{i j} \ll 0$ ), but if a vote is only likely to be lost-perhaps as with a confusing ballot design - then $-\infty<L_{i j}<0$. Lost votes present two primary challenges for the eforensics model. Because election administrative failures are often localized, $L_{i j}$ is likely to be nonzero only for electors in some precincts: sometimes almost all voters in some precincts have $L_{i j}<0$ while all or almost all voters elsewhere have $L_{i j}=0$. Using $L_{i}$ to denote the precinct average of $L_{i j}, L_{i}$ induces dependence patterns similar to those induced by $D_{i}$ : for pairs of precincts $i$ and $i^{\prime}$ that have lost votes, $\operatorname{cov}\left(L_{i}, L_{i^{\prime}}\right) \neq 0$.

Such association can induce false positive eforensics-frauds estimates. But because the eforensics specification includes only eforensics-frauds where votes are added for the leader - see (2a), (2b), $a_{i}^{*}$ and $w_{i}^{*}$ - the specification cannot faithfully represent lost votes that subtract voters or subtract votes. So false positive eforensics estimates will alias the reductions due to lost votes in unknown ways through the parameters and estimated eforensics-frauds.

Of course votes might be lost due to malevolent actions, for example voter suppression or spoiling the votes for a candidate. In such cases the appropriate idealized individual-level specification for the malevolent distortions is (8) and not (6), but a problem remains because the eforensics specification cannot faithfully represent lost votes that subtract voters or votes. What parameter values will most closely approximate the data in such situations is unclear.

An important nuance is that if votes intended for a non-leading candidate are lost more often than are votes intended for the leader, then it's as if a positive $D_{i}$ term has been included in (6) corresponding to the individual-level leader vote choice probability: if the
lost votes are lost due to malevolent actions then the resulting estimated eforensics-frauds may reflect the malevolent actions but they do not literally represent vote counts that were added to the leader; instead, votes the non-leading candidate lost may appear as eforensics-fraudulent votes gained by the leader. In such cases we expect eforensics estimates to feature aliased representations of the vote-losing malevolent distortions.

While it is unclear in general what patterns aliased eforensics estimates induced by lost votes will have, by using multiple chains in the MCMC algorithm we may observe symptoms that suggest that lost votes are present. If the functional form of the likelihood used for a Bayesian estimation does not closely approximate the process that generated the data, the result is often multimodality in the posterior distribution of the Markov chains (cf. Grün and Leisch 2009): the algorithm tries various parameter estimates to coerce the model to approximate as closely as it can the process the model does not really match; usually there are several such rough approximations, each of which is associated with a local mode of the posterior distribution. Because the eforensics specification includes only eforensics-frauds where votes are added for the leader-again see (2a), (2b), $a_{i}^{*}$ and $w_{i}^{*}$ - the model specification cannot faithfully represent lost votes. If lost votes occur we therefore expect to observe posterior multimodality. Conversely, if we observe posterior multimodality that is evidence that there are lost votes, although of course there may other sources of model misspecification. If we run the MCMC algorithm for a sufficient number of iterations with a single chain for such a misspecified model the chain should eventually exhibit posterior multimodality, but in practice it is more effective to use several chains. ${ }^{11}$ To be determined is how substantial the posterior multimodality needs to be to support an inference that there are lost votes: we will draw on empirical evidence to understand how posterior multimodality in the mixture probabilities relates to the occurrence of lost votes.

[^8]Multiple Ambiguities For some elections there will be precincts that combine variants of all the additions $D_{i}, S_{i j}$ and $L_{i j}$ in (6), (7) and (8), so the sources of ambiguity in eforensics estimation may compound or interfere with one another. In general there is no reason to expect the various sources of ambiguity will cancel one another. Estimates from eforensics may therefore require nuance and care to interpret.

## 3 Lost Votes (and Some Strategic Behavior)

One implication of the notional individual-level constructions of section 2.3 is that malevolent distortions of elector intentions, strategic elector behavior and lost votes can all prompt dependencies among individual electors that aggregate into dependencies (or associations or correlations) among aggregation units such as polling stations or precincts. While the specific forms such associations may take is unclear, as a general matter we can say that distributions of aggregation unit proportions will be clumpy. While illustrating such clumpiness in data from several elections, I illustrate how in eforensics model estimates dependencies that appear as clumpiness can produce measurable multimodality in MCMC posterior distributions for mixture probabilities. Particularly I suggest that posterior MCMC multimodality is an indication that there are lost votes, although lost votes do not necesarily trigger such multimodality and other kinds of model misspecification can also trigger such multimodality. For example posterior MCMC multimodality can appear when the Normal priors used for the frauds magnitude parameters are insufficient.

### 3.1 Clumps and Entropy

Associations among aggregation units will generally manifest as clumpiness. Plots such as the eforensics-plots shown in Figure 2 depict such clumpiness in a scatterplot of turnout proportions by the proportions of votes cast for the leader. The figure shows polling station
results from the 2009 president election in Afghanistan, where Hamid Karzai received the most votes (Democracy International 2009) and is treated as the eforensics leader. The eforensics-plot shows a scatterplot with histograms along the margins and a two-dimensional empirical density behind the scatterplot. The original data in Figure 2(a) exhibit turnout proportions that range from extremely low to 1.0 , and proportions of votes for the leader that range from 0.0 to 1.0. Because tribal and other bases for candidates' political support varies regionally in Afghanistan, Figure 2(b) displays the data after province fixed effects are removed. Clumps of points are apparent in both plots.

The eforensics-plots shown in Figure 3 show data from the Uganda 2011 president election, where the leader is Yoweri Museveni. Figure 3(a) exhibits a feature that Klimek et al. (2012) emphasized as a "fingerprint of fraud": the original data feature a concentration of points in the upper-right corner of Figure 3(a) that have very high turnout and very high proportions of votes for the leader. But regional variations in the candidates' support motivates removing province fixed effects, which produces Figure 3(b) in which such "fingerprints" are not apparent. Nonetheless clumps of points are apparent in both plots.

To make some of the clumps in the Afghanistan 2009 and Uganda 2011 plots easier to see, Figure 4 magnifies the top-right portions of Figure 2(b) and Figure 3(b). ${ }^{12}$ Clumps are easy to see in Figure 4: perhaps the Afghanistan data appear to be clumpier than are the Uganda data.

Tables 3 and 4 display eforensics-model estimates for the Afghanistan 2009 and Uganda 2011 president elections. For Afghanistan the high number of polling stations with extreme frauds and the occurrence of a positive value for the incremental frauds magnitude parameter $\rho_{M 0}$ are strong indications that malevolent distortions of electors' intentions

[^9]occur. Removing $F_{t}$ and $F_{w}$ would leave the leader with vote proportion
$$
\frac{3093256-647006.3}{5662758-512311.2}=.475
$$
less than the threshold of .5 needed to avoid a runoff election (Electoral Complaints Commission 2010, 37). For Uganda extreme frauds are ample if not potentially decisive: the estimated proportion of leader votes that are eforensics-fraudulent is $312556.3 / 5436639=.0575$; the number of eforensics-fraudulent votes is not greater than the margin of $5436639-2071397=3365242$ between first and second. Note that the Uganda eforensics model specification includes province fixed effects for turnout, vote choice and eforensics-frauds magnitudes parameters that are not shown.

Most relevant for a focus on posterior MCMC multimodality and lost votes are two diagnostics for posterior multimodality. Both Tables 3 and 4 report for the mixture probability parameters dip tests of unimodality (Hartigan and Hartigan 1985) (denoted $\left.D\left(\pi_{j}\right), j=1,2,3\right)$ and computations of the differences among the posterior means across the four MCMC chains (denoted $\left.M\left(\pi_{j}\right), j=1,2,3\right)$. For $D\left(\pi_{j}\right)$ a unimodality null hypothesis is tested for the combination of all chains for each $\pi_{j} . M\left(\pi_{j}\right)$ reports the largest absolute difference between pairs of chain-specific posterior means. In Table $3 D\left(\pi_{1}\right)$ and $D\left(\pi_{2}\right)$ are significant (the $p$-values are effectively zero), and $M\left(\pi_{1}\right)$ and $M\left(\pi_{2}\right)$ have values that we will see are too large to ignore. In Table 4 no dip test value is significant but again $M\left(\pi_{1}\right)$ and $M\left(\pi_{2}\right)$ have values that we will see are large. Our interpretation is that probably both elections feature lost votes, or perhaps other misspecifications, even though the eforensics estimates also suggest both elections include ample manufactured votes. It is remarkable that the model for Uganda features posterior multimodality even though the model specification includes province fixed effects for turnout, vote choice and eforensics-frauds magnitudes. ${ }^{13}$

[^10]"Looks clumpy" is not a precise criterion for assessing clumpiness hence underlying dependence among individual electors, so to measure clumpy dependence among aggregation units we compute the entropy of turnout and vote proportion scatterplots. We base our entropy measure on the cell counts produced by fitting each scatterplot into a $1000 \times 1000$ grid, computing the proportion $q_{j k}$ of points in each cell. "Gridded" entropy is then $E=-\sum q_{j k} \log \left(q_{j k}\right)$. The corresponding efficiency is $E / \log \left(N^{+}\right)$where $N^{+}$is the number of nonempty cells. Tables 5 and 6 display such entropy calculations for the Afghanistan 2009 and Uganda 2011 data along with data from several other elections. ${ }^{14}$ I compute entropy values after any fixed effects are removed ("residualized" observed entropy). The tables also report Normal simulated data entropy values produced by computing gridded entropy for observations simulated to have the same mean and covariance as the residualized data (observed minus fixed effects), using the same number of observations as in the observed data.

The most important result in Tables 5 and 6 is that for every election the residualized observed entropy is less than is the Normal simulated entropy. All of malevolent distortions of elector intentions, strategic elector behavior and lost votes - and other things - can trigger clumpiness, as discussed in section 2.3. The entropies suggest that one or more of these occur in all of the elections included in the tables. While entropies are not comparable across elections, the efficiency values are. The efficiency for Afghanistan 2009 is smaller than the efficiency for Uganda 2011: . 9611 versus .9947. Only one election in the tables-Peru 2021 President Round 1—has lower efficiency than does Afghanistan 2009. Several ${ }^{15}$ have efficiencies higher than that of Uganda 2011. In work reported here and elsewhere, I find that the efficiency values are not correlated with the extent or magnitude of eforensics-frauds. Clumpiness is pervasive but not necessarily eforensics-frauds, let alone malevolent distortions of elector intentions. Only clumpiness that connects

[^11]appropriately with the functional forms of the eforensics model specification and its estimator leads to nonzero estimated eforensics-frauds. ${ }^{16}$ The question is whether and when such eforensics-frauds are measuring malevolent distortions as opposed to the several sources for ambiguities.

Two elections illustrate a common pattern in which different kinds of polling stations are associated with aggregate dependencies (or "clumps"). Figure 5 shows poll data eforensics-plots for the Canada 2011 federal election. Data from all districts are pooled: the leader in each district is the candidate with the most votes. In Figure 5(a) the original data show that a notable proportion of polls have turnout at or near 1.0. These polls are Mobile polls and Special Voting Rules (SVR) polls (Elections Canada 2022). This turnout-extreme clump migrates to the middle of the scatterplot when the data are adjusted for district and poll-type fixed effects. Figure 6 shows casilla (ballot box) data for the Mexico 2006 president election. High turnout is apparent among what are mostly casillas especiales. ${ }^{17}$ Figure 7 displays the tops of the two elections' residualized data scatterplots. Clumps are apparent in both, although more so for Canada 2011. In Table 5 notice that the efficiency for Canada 2011 is lower than that for Mexico 2006 President.

Last in this set is the Russia 2011 Duma election (polling stations shown in Figure 8). ${ }^{18}$ The leader is the United Russia party. A feature that will receive more attention in section 3.3 is the clump at the bottom near the middle of the residualized-data plot in Figure 8(b). We will see that many of those low-valued polling stations have an experimentally driven reason for their common relatively low values.

### 3.2 Lost Votes and MCMC Multimodality

To help gauge how lost votes manifest in eforensics estimates I turn first to a collection of elections that can reasonably be expected not to suffer from malevolent distortions of

[^12]electors' intentions (perhaps other than misinformation efforts) and that are likely minimally affected by strategic elector behavior (perhaps other than strategic abstention). I examine how the posterior multimodality measures $D\left(\pi_{j}\right)$ and $M\left(\pi_{j}\right)$ values are related to participation and imbalances in these elections, then how these features are associated with eforensics-frauds. Then I expand the scope of the assessments to a collection of legislative elections. Next I exploit an example in which lost votes were induced by a field experiment. The section concludes by considering several examples, one in which lost votes can easily be seen in scatterplots and then several in which lost votes are notorious and arguably decisive for the election outcomes.

Ballot Propositions The elections I study first are votes regarding U.S. state ballot propositions, i.e., constitutional amendments, referenda and the like. An example of propositions from California in 2008 is shown in Table 7. For the twelve propositions apppearing on the ballot that year, the table reports for each proposition the total number of votes cast on the proposition (summing all YES or NO votes), the proportion of all votes cast for any office or item that were cast for the proposition, and the proportions of votes cast for the proposition that are either YES or NO votes. The last two columns in Table 7 contain respectively $\min _{j \in\{1,2,3\}} D\left(\pi_{j}\right)$ and $\max _{j \in\{1,2,3\}} M\left(\pi_{j}\right)$. Participation varies across propositions, ranging from a low of .87 for proposition 11 to a high of .98 for proposition 8 . The asymmetry of support versus opposition for propositions varies: the most lopsided vote is that for proposition $6(.28$ YES, .62 NO$)$ while the closest vote is for proposition 11 (. 44 YES, .43 NO).

Tables 8 and 9 show that the diversity of posterior multimodality measures displayed in Table 7 is matched by a diversity of eforensics-fraudulent votes estimates. The leader for each of the eforensics models is the alternative that has the most votes. Both the number of eforensics-fraudulent precincts and of eforensics-fraudulent votes varies considerably across propositions. For example proposition 9 has no eforensics-fraudulent
precincts while for proposition 710544 precincts are eforensics-fraudulent. Proposition 9 has no eforensics-fraudulent votes while proposition 7 has $F_{w}=681066.7$. Continuing the assumption that in these proposition votes there are no (or at least only scant) malevolent distortions of electors' intentions, what explains the great diversity of eforensics-frauds estimates?

To address that question I bring together the 75 propositions from elections in several states summarized in Table 10. Figure 9 shows scatterplots of the state-level proportions and $M\left(\pi_{2}\right)$ values for these propositions: the $x$-axis is either (a) the proportion of votes cast that were either YES or NO or (b) the maximum of either the YES or NO proportions divided by the proportion voting either YES or NO; ${ }^{19}$ the $y$-axis is $M\left(\pi_{2}\right)$. Overall participation (Figure $9(\mathrm{a})$ ) is not related to $M\left(\pi_{2}\right)$, but $M\left(\pi_{2}\right)$ tends to increase as the vote becomes more lopsided (Figure 9(b)). Regression model parameter estimates in Table 11 further support the idea that lopsidedness is related to $M\left(\pi_{2}\right)$ but the mere proportion participating per se is not. Using only the maximum proportion as regressor (Table 11(2)) is better than including the proportion voting. Such a result is not surprising: lower participation that reduces both alternatives symmetrically should produce a more negative estimate for $\beta_{0}$ hence smaller $\tau_{i}$ but not otherwise much distort the eforensics model. But nothing about the model specification allows it faithfully represent asymmetric declines in participation. Recall the discussion of $L_{i j}$ and (8) in section 2.3: asymmetric declines in participation in the ballot proposition elections are lost votes in the spirit of that discussion.

Posterior MCMC multimodalities also associate with the frequency and magnitude of eforensics-frauds, although the relationships are not simple. We can show some of the complexity of the relationships but by no means can we claim to have adequately described them. Figure 10 presents the first part of that effort: for each proposition $F_{w} / \sum V_{i}$ is plotted against $M\left(\pi_{2}\right)$. Clearly there is an increasing relationship, but as well it is clear

[^13]that the relationship is not well-described as linear. Figure 11 breaks $F_{w}$ into its four natural components ((a) incremental $F_{t}$, (b) incremental $F_{w}-F_{t}$, (c) extreme $F_{t}$ and (d) extreme $F_{w}-F_{t}$ ). In most of the scatterplots there is an increasing relationship but again such a relationship does not very well characterize how the multimodality measure relates to the eforensics-frauds.

The multinomial logit (MNL) model estimates in Table 12 explain part of the reason simple linear characterizations of the relationship between $M\left(\pi_{2}\right)$ and $F_{w} / \sum V_{i}$ or its components fail. Beyond using a model that appropriately represents having count data, ${ }^{20}$ the results show that not only is $D\left(\pi_{2}\right)$ independently associated with the magnitude of eforensics-frauds but the multiplicative interaction $D\left(\pi_{2}\right) \times M\left(\pi_{2}\right)$ has very strong effects.

Legislative Elections I now apply a similar form of analysis to data from several single-member district (SMD) legislative elections. The elections are from Bangladesh, Canada, Germany and Mexico. To start I show eforensics-plots and eforensics model estimates for at least one of the elections from each of the four countries.

Data for Erststimmen from the 2021 Bundestag election are shown in Figure 12. The leader in each Wahlkreis (district) is the party with the most votes there. The clump of polling stations with extremely low turnout seen in Figure 12(a) are for Briefwahl: relative to the often very large sets of electors who are eligible to vote in them, such polling stations usually exhibit very low turnout. In Figure 12(b) where fixed effects both for Wahlkreis and Wahlbezirk (polling station) type are removed the clump of Briefwahl polling stations is shifted from the extreme to near the middle of the turnout distribution. The eforensics specification used to produce the estimates reported in Table 13 includes the same kinds of fixed effects for turnout and vote choice used for Figure 12(b). Even though the number of eforensics-frauds is small, both the fact that incremental frauds occur while $\rho_{M 0}$ and $\rho_{S 0}$ have indeterminate signs and that posterior MCMC

[^14]multimodality occurs $\left(D\left(\pi_{2}\right)=0\right)$ suggest there were problems in the election. One possibility is irregularities that beset voting in Berlin (Wikipedia 2023).

Earlier Bundestag elections exhibit more considerable measures of posterior MCMC multimodality. For example for Erststimmen in 2005, shown in eforensics-plots in Figure 13 , eforensics estimates (Table 14) have not only $D\left(\pi_{2}\right)=0$ but $M\left(\pi_{2}\right)=.0179$. $F_{w}=360137.2$ is greater than in 2021. The incremental frauds for 2005 are arguably less problematic than they are in 2021, even though they are more numerous (360069.4 versus 15031.5) because in $2005 \rho_{M 0}$ and $\rho_{S 0}$ are strictly negative while in 2021 they are not: strictly negative incremental frauds magnitude parameters often occur when only strategic elector behavior has occurred, without any malevolent distortion of electors' intentions; even if only due to wasted-vote considerations, strategic behavior occurs in Erststimmen votes (e.g. Harfst, Blais and Bol 2018). In 2005 election administration problems caused votes to be lost in Dortmund, with consequences that appear to be administratively less adverse than what is occurring now in Berlin as a result of 2021 (Bundestag March 3, 2006; Mebane and Klaver 2015).

Table 15 reports eforensics estimates for the Canada 2011 data displayed in Figure 5. The model specification includes the same fixed effects for turnout and vote choice that were used to produce Figure 5(b), as well as a fixed effect for polls that needed to have their counts of electors adjusted. ${ }^{21}$. Even though $F_{w}=33238.6$ is small in comparison to $\sum_{i=1}^{n} W_{i}=7307339$, worrisome in these results is $\rho_{m 0}>0$ in addition to the occurrence of polls that have extreme frauds. Also $D\left(\pi_{2}\right)=0$, even though $M\left(\pi_{2}\right)$ is too small to signal concern about posterior MCMC multimodality.

An optimistic view is that the concerns raised by the eforensics estimates for Canada 2011, which resemble similar problems estimated using data from the 2004, 2006, 2008 and 2015 Canada elections, are all due to lost votes. The 2011 election was marked by a robocall scandal in Guelph (Devlin 2012) that may have contributed to lost votes there in

[^15]that election. ${ }^{22}$ But Table 16 suggest something more systematic is going on. The table reports a breakdown by administrative type of the eforensics-fraud type of each poll, pooling over the five federal elections during 2004-2015: what proportion are classified as having no frauds, incremental frauds or extreme frauds. While regular residential polls have very small frequencies of incremental or extreme frauds, the frequency of incremental frauds for Mobile polls is seven times larger while the frequencies for SVR polls are more than four times greater. Extreme frauds occur for Mobile and SVR polls with frequencies that range from two to twelve times larger than are the frequencies for regular residential polls. It is preferable for these discepancies to be due to differing occurrences of lost votes rather than to some kind of malevolent distortions of electors' intentions.

For Mexico we have the Mayoria Relativa casilla votes in the 2006 Deputies election. The leader in each district is the candidate with the most votes there. Figure 14 shows eforensics-plots. These resemble the data shown for the 2006 president election (Figure 6) in several respects, and the 2006 Deputies and president elections have similar entropies (Table 5). Table 17 reports eforensics estimates. The model specification includes district fixed effects for turnout and vote choice. About 1.5 percent of leaders' votes are eforensics-fraudulent: $F_{w}=186060.1$ out of $\sum_{i=1}^{n} W_{i}=11914080$ votes. That both $\rho_{M 0}<0$ and $\rho_{S 0}<0$ suggests that many of the eforensics-fraudulent votes may be due to strategic elector behavior: wasted-vote behavior should be expected given the presence of, generally, three dominant parties or coalitions plus two smaller parties (Klesner 2007). Neither $D\left(\pi_{2}\right)$ nor $M\left(\pi_{2}\right)$ have values that suggest there is posterior MCMC multimodality.

For Bangladesh we have the 2001 election, which produced such controversy that a losing party boycotted the subsequent legislature (European Union 2001; Centre for Research and Information 2002). The leader in each district is the candidate with the most votes there. Figure 15 shows eforensics-plots. Specifications for the eforensics estimates reported in Table 18 include district fixed effects for turnout, vote choice and

[^16]eforensics-frauds magnitudes. With these fixed effects included, neither $D\left(\pi_{2}\right)$ nor $M\left(\pi_{2}\right)$ have values that suggest there is posterior MCMC multimodality, but if the frauds magnitudes fixed effects are omitted then $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.0967$ : what might appear to be evidence of lost votes instead appears to reflect that in this case to represent the frauds magnitude parameters it is insufficient to rely on only intercepts in $x_{i}^{\iota}$ and $x_{i}^{v}$ along with the Normal priors for $\kappa_{i}^{\iota M}, \kappa_{i}^{\iota S}, \kappa_{i}^{v M}$ and $\kappa_{i}^{v S}$ in (4)(c) and (4)(d). That both $\rho_{M 0}<0$ and $\rho_{S 0}<0$ suggests that many of the eforensics-fraudulent votes may be due to strategic elector behavior: both wasted-vote and coalition behavior should be expected given the large number of parties in the election (Bangladesh Election Commission 2002, 15 , lists 54 parties), including an important four-party coalition. But extreme eforensics-frauds are ample: 307 polling stations are classified as having extreme frauds, and votes that have extreme frauds total $F_{w}=273598.0$. Overall nearly five percent of leader votes are eforensics-fraudulent: $F_{w}=1387497.6$ out of $\sum_{i=1}^{n} W_{i}=28967523$.

Figure 16 displays the frauds magnitude fixed effects that are "active" in the sense that they are associated with at least one polling station that is classified by the model of Table 18 as eforensics-fraudulent. Almost every district has an active fixed effect for incremental frauds magnitudes, and most have active fixed effects for extreme frauds. Taking into account the boundaries of the fixed effects' credible intervals, several districts are diverse in the sense that they have fixed effects that differ significantly in size from one another. ${ }^{23}$

The eforensics-frauds estimated using the estimates in Table 18 are big enough to have changed election outcomes. As Table 19 shows, in nine districts $F_{w}$ is bigger than $M$, the margin between the first-place and second-place candidates. Removing the eforensics-frauds from the leader's votes, with no other changes, would have put the

[^17]second-place candidate into the lead.
Having illustrated some of the main features of legislative elections from Bangladesh, Canada, Germany and Mexico, I return to the task of evaluating how the two kinds of measures of posterior MCMC multimodality associate with eforensics-frauds. Overall I include data from 15 elections: Bangladesh 1991, 1996, 2001; Canada 2004, 2006, 2008, 2011, 2015; Germany 2002, 2005, 2009, 2021; Mexico 2006, 2009, 2012. For several of the assessments I use $D\left(\pi_{2}\right)$ and $M\left(\pi_{2}\right)$ values taken from "nonpooled" eforensics estimates: I estimate the model separately for each district. Estimates such as I displayed in Tables 1, $13,14,15,17$ and 18 are "pooled": data from all districts are included in the same one eforensics model with district fixed effects being used at least for turnout and vote choice. As I discuss elsewhere, these approaches produce similar but not identical results.

Figure 17 shows scatterplots of $M\left(\pi_{2}\right)$ and $D\left(\pi_{2}\right)$ for all districts in each country over the included years. As should be expected, $D\left(\pi_{2}\right)$ is small-indeed $D\left(\pi_{2}\right)=0$-when $M\left(\pi_{2}\right)$ is large, but a bit surprising is that $D\left(\pi_{2}\right)>0$ even when $M\left(\pi_{2}\right)$ is not small. For instance, for Bangladesh (Figure 17(a)) $D\left(\pi_{2}\right) \approx .8$ while $M\left(\pi_{2}\right) \approx .3$. A range of $D\left(\pi_{2}\right)$ values occurs corresponding to quite small values of $M\left(\pi_{2}\right)$ : e.g., $D\left(\pi_{2}\right) \geq 0$ for $M\left(\pi_{2}\right) \geq .0143$ in Germany and for $M\left(\pi_{2}\right) \geq .00919$ in Mexico. Such small values of $M\left(\pi_{2}\right)$ that correspond to $D\left(\pi_{2}\right)=0$ are one reason I said in relation to Table 3 and subsequently that even a value of $M\left(\pi_{2}\right)$ as small as .01 is too large to ignore as a signal for potential posterior MCMC multimodality.

The other reason not to ignore otherwise small $M\left(\pi_{2}\right)$ values is a set of binomial regression models in which eforensics-fraudulent classifications or votes are the outcome variables and $M\left(\pi_{2}\right)$ and $D\left(\pi_{2}\right)$ are regressors. Table 20 reports estimates for such regression models that have as outcomes counts of polling stations in each district that are classified as either having or lacking eforensics-frauds using nonpooled (district-specific) eforensics estimates. In the models the reference category is "no frauds." The regression models include fixed effects for each election. As in the model for eforensics-frauds
among ballot propositions (Table 12), a multiplicative interaction between $M\left(\pi_{2}\right)$ and $D\left(\pi_{2}\right)$ substantially improves the model: excluding the interaction (Table 20(a)) is clearly inferior to including it (Table 20(b)). Particularly if $D\left(\pi_{2}\right)>0$, a small value of $M\left(\pi_{2}\right)$-versus $M\left(\pi_{2}\right)=0$-can be associated with an importantly large increase in the proportion of polling stations that are classified as eforensics-fraudulent.

Table 21 shows that a similar situation holds for the magnitude of eforensics-frauds, hence for the number of eforensics-fraudulent votes, $F_{w i}$. Table 21(a,b) reports binomial regression results for pooled estimates, and Table 21(c,d) reports results for nonpooled estimates. The pooled and nonpooled estimates are qualitatively similar. While the coefficients for the interaction are smaller than in the model for eforensics-frauds' occurrences, the effects nonetheless imply that seemingly small values of $M\left(\pi_{2}\right)$ should not be ignored, particularly when $D\left(\pi_{2}\right)>0$. A little posterior MCMC multimodality can go a long way.

Lost votes are a primary reason for eforensics estimates to exhibit posterior MCMC multimodality, but as I've mentioned lost votes are not the only reason. Several other elections resemble Bangladesh 2001 in that signs of posterior MCMC multimodality disappear once geographically defined fixed effects are included for the eforensics-frauds magnitude parameters. Other features of the eforensics model specification may also be erroneous hence prompt signs of posterior MCMC multimodality.

But the occurrence of posterior MCMC multimodality need not doom the model. Recall that the eforensics model for the 2017 National Assembly election in France, for which there is strong correspondence with district annulment decisions by the Conseil Constitutionnel, has $D\left(\pi_{2}\right)=0$. In that case $M\left(\pi_{2}\right)$ is probably too small to matter. In that case - if the decisions by the Conseil Constitutionnel can be trusted (as the French public generally seems to do (Klaver 2023))—diagnosed posterior MCMC multimodality does not prevent valid measurement of malevolent distortions of electors' intentions.

Moreover even if there are signs of posterior MCMC multimodality more substantial
than occur in the 2017 election in France, eforensics estimates can provide important insights. I'll consider examples of this in sections 3.4 and 3.5 below.

### 3.3 Lost Votes and an Experiment

An election observation field experiment in Moscow during the 2011 Russia Duma election provides data that can help confirm aspects of the lost votes mechanism. Enikolopov, Korovkin, Petrova, Sonin and Zakharov (2013) randomly assigned observers to polling stations on election day in Moscow, finding that the presence of observers at a polling station reduced both reported turnout and the reported vote for the United Russia party. They estimated treatment effects in the same directions but a bit attenuated at neighboring polling stations. I use these experimentally induced reductions in turnout and United Russia vote choices as implementations of a kind of lost votes. These are lost votes for which, due to the experimental design, the vote-losing mechanisms are transparent.

Table 22 reports eforensics estimates for 2011 Duma election including data only from Moscow. ${ }^{24}$ The leader is United Russia. When the model is estimated for all of Russia with region fixed effects included for turnout and vote choice, no eforensics-frauds occur in Moscow. Elsewhere I discuss this pattern in which eforensics typically underestimates malevolent distortions of elector intentions in Russia (there are "too many frauds"). But Table 22 shows that when Moscow votes are treated separately eforensics estimates are notable: $F_{w}=51643.9$ out of $\sum_{i=1}^{n} W_{i}=2052751$, which is $2.5 \%$ of leader votes. To many that number seems low, and it is smaller than the number estimated by Enikolopov et al. (2013). $\rho_{M 0}>0$ and there are many extreme frauds with $\delta_{M 0}>0$ : it is not surprising to find evidence of manufactured votes (Arbatskaya 2004). $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.0495$, so there are clear signs that posterior MCMC multimodality occurs.

The display in Figure 18 displays scatterplots of the original data. The
eforensics-plot in Figure 18(a) shows a number of distributional irregularities, but the

[^18]most important thing to recognize is that the Moscow polling stations are frequently among the polling stations represented in the clump at the bottom near the middle of the residualized-data plot in Figure 8(b). Figure 18(b) draws in blue ' $x$ ' characters the polling stations Enikolopov et al. (2013) experimentally assigned to have observers, in red triangles the polling stations that have eforensics-frauds and in green circles the remaining polling stations. Clearly the experimentally observed polling stations have lower turnout and vote proportions for the leader than do the rest of the polling stations. Moreover no experimentally observed polling stations is eforensics-fraudulent. Figure 19 additionally uses distinct characters for the neighboring polling stations: neighbors appear either as tan crosses or red asterisks; the latter are eforensics-fraudulent. Neighors of experimentally observed polling stations mostly but not entirely avoid having eforensics-frauds.

The experiment induces lost votes that primarily reduce the leader's support. For the eforensics estimates reported in Table 23, I included "is observed" and "is neighbor" dummy variables as regressors for turnout and vote choice. Both variables have negative coefficients in both equations, which reaffirms the findings of Enikolopov et al. (2013). Beyond these reductions in leader support, eforensics finds $F_{w}=65338.9$ : a larger number of eforensics-frauds are estimated when the experimental observation design is taken into account than when it is not; the $99.5 \%$ credible interval for $F_{w}$ in Table 23 is strictly greater than is the corresponding interval in Table 22 . Still $\rho_{M 0}>0$ even if now $\delta_{M 0}$ has an indeterminate sign.

Most important for the purpose of understanding posterior MCMC multimodality and lost vote mechanisms, in Table $23 D\left(\pi_{2}\right) \gg 0$ and $M\left(\pi_{2}\right)$ is probably too small to matter. Taking the experiment into account in turnout and vote choice-hence accurately accounting for the principal source of lost votes - apparently eliminates the posterior MCMC multimodality.

### 3.4 Lost Votes in Argentina 2015 President Elections

The 2015 president election in Argentina, which required two rounds of voting, exhibits lost votes in an unusual but easy to understand way. Lost votes that affect eforensics appear for only one the rounds.

Figure 20 shows eforensics-plots for mesa data from the first round. Voting is compulsory in Argentina, but from the original data in Figure 20(a) it is obvious that not everyone votes. Blank votes are included as votes cast, and the leader is the candidate with the most votes (Daniel Scioli of Frente para la Victoria, FPV). In Figure 20(b) departamento fixed effects are removed. A trailing of mesas that have very low turnout even when departamento means are removed is apparent.

Zhang, Alvarez and Levin (2019) use a rich collection of covariates in simulation and random forest methods to study anomalies in the first round election. They find only $86.3 \%$ of mesas are "Clean."

Estimates using eforensics imply that eforensics-frauds in the first round are scant. Table 24, using a model specification without fixed effects, reports $F_{w}=2894.2$ out of $\sum_{i=1}^{n} W_{i}=9002242: 2894.2 / 9002242=.000321$. Both $\rho_{M 0}$ and $\rho_{S 0}$ have indeterminate signs, but only 31 of 92204 mesas have incremental frauds. Only 10 mesas have extreme frauds. Both because Zhang, Alvarez and Levin (2019) used several covariates and because posterior MCMC multimodality is apparent- $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.0177$ - there are strong reasons to include geographic fixed effects. Demographic variables may work better, but elsewhere I discuss the risks of using covariates that match aggregation unit behavior too closely.

The 99 departamentos I distinguish to define fixed effects for the eforensics specification used in Table 25 seem reasonably sufficient to reveal whether including demographic or other variables would greatly change the results. ${ }^{25}$ Table 25 reports that

[^19]with departamento fixed effects for turnout and vote choice eforensics-frauds do increase but only to $F_{w}=6326.1$. The number of eforensics-fraudulent votes nearly doubles when compared to the specification that omits fixed effects, but still the estimate is well short of what Zhang, Alvarez and Levin (2019) find: $6326.1 / 9002242=.000703 ; \varphi=.00168$. Now both $\rho_{M 0}<0$ and $\rho_{S 0}<0$, which suggests that many of the incremental frauds may stem from strategic elector behavior: there were six candidates with a majority rule, and the top three received proportions $.21, .34$ and .37 of the votes; the outcome resembles Bouton and Gratton (2015)'s "Duverger's hypothesis equilibrium." Only 5 mesas have extreme frauds. That $D\left(\pi_{2}\right)=1$ and $M\left(\pi_{2}\right)=.0015$ suggests there is not posterior MCMC multimodality, hence there is little reason to doubt the adequacy of the eforensics model specification used to produce Table 25's estimates.

The eforensics-plots for round 2, shown in Figure 21, display a tail of mesas that have very low turnout that is more extensive than occurs for round 1 . Despite the apparent expansive spread of those mesas, it is important to notice the implication of the marginal histogram at the top of each graph: only a small proportion of the mesas have such low turnout; precisely, 2014 of the 92632 mesas have a turnout proportion less than .6. Nonetheless are those mesas asymmetric lost votes-perhaps involving would-be supporters of opposition candidates more than of the leader - of a kind that triggers posterior MCMC multimodality? The leader is the party that received the most votes, Mauricio Macri of Cambiemos.

Table 26 reports eforensics for a model specification that omits fixed effects (Table 26(a)) and one that includes departamento fixed effects for turnout and vote choice (Table 26(b)). Interestingly the specification that lacks fixed effects also lacks signs of posterior MCMC multimodality, while the model that includes fixed effects has $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.0913$. For both specifications the eforensics-frauds are few and small: Table 27 reports no incremental frauds and scant extreme frauds, with very small $F_{w}$ values.

That apparent posterior MCMC multimodality increases when fixed effects are specified
for turnout and vote choice suggests fixed effects ought to be applied as well to the eforensics-frauds magnitudes parameters. As reported in Table 28, with that change the estimates change considerably. Now more mesas have eforensics-frauds than have no frauds, even though $\pi_{1}=.538$, and $F_{w}=1450675.7$ (out of $\sum_{i=1}^{n} W_{i}=12711629$ ). Signs of posterior MCMC multimodality are ample: $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.152$.

Strategic abstention might explain the estimates. Strategic abstention can generate asymmetric lost votes hence posterior MCMC multimodality. By this explanation the long trail of low turnout mesas in Figure 21 is only the most visible symptom of coordinated abstentions that pervade the electorate, especially abstentions by would-be opposition supporters. At least, that seems to be the simplest explanation, supported by Figure 22 which shows that incremental eforensics-frauds and their fixed effects are active in almost all departamentos, and a few $\rho_{M j}$ values have indeterminate signs. ${ }^{26}$ Extreme frauds are both rarer and less diverse. The eforensics model is misspecified, but nonetheless the estimates reveal a potentially interesting feature of the election, one that may help to explain the election's otherwise surprising outcome, even though assessment solely of malevolent distortions of electors' intentions is made more difficult.

### 3.5 Lost Votes in Three U.S. President Elections

At least three recent elections for president in the United States have featured consequential lost votes. Of the three I'll briefly discuss in this section, two elections were arguably decided for the whole country by lost votes and for the third the outcome in a key state was decided by lost votes. The three examples are Florida 2000, Ohio 2004 and Wisconsin 2016.

Florida 2000 In 2000 in Florida incompetent election administration caused tens of thousands votes to be lost (Wand, Shotts, Sekhon, Mebane, Herron and Brady 2001;

[^20]Wolter, Jergovic, Moor, Murphy and O'Muircheartaigh 2003; Mebane 2004), and there were efforts to suppress voters (Berman 2015). With an official margin of 537 votes in favor of the leader (George W. Bush, Republican), almost every one of the administrative failures in that election was decisive in producing the wrong winner, because most of the failures most adversely affected electors who supported opposition. In the eforensics-plots shown in Figure 23, the display of the original data reveals a tendency for precincts with lower turnout to vote less favorably for the leader than do precincts with higher turnout. The plot with county fixed effects removed (Figure 23(b)) features readily visible clumps, even though among the elections with entropy shown in Tables 5 and 6 , the Florida 2000 president election has relatively high efficiency.

Estimates for the eforensics model reported in Table 29 show that eforensics-fraudulent votes greatly exceed the official margin of victory: $F_{w}=85359.7$. The model specification includes county fixed effects for turnout, vote choice and eforensics-frauds magnitudes. Signs of posterior MCMC multimodality persist: $M\left(\pi_{2}\right)=.0155$. Only incremental frauds are active, and Figure 24 displays eforensics-frauds magnitudes fixed effects. These fixed effects are active for most counties and are diverse. Notably the third greatest fixed effect (posterior mean) for manufactured votes and the greatest for stolen votes are in Duval County, where among all Florida counties the highest proportion of votes were lost due to a deficient (two-page) ballot design (Mebane 2004): this illustrates how eforensics appears to represent votes lost by opposition as votes gained by the leader; the model aliases the lost votes. There is no evidence that election processes actually added votes for Bush but plenty that opposition-and particularly Democrat Al Gore - lost more votes than did Bush. Ten candidates were on the ballot and both $\rho_{M 0}<0$ and $\rho_{S 0}<0$, so it is likely that strategic elector behavior (particularly wasted-vote behavior and mobilization) contribute to the estimated incremental frauds, along with the lost votes, although to what extent we cannot say.

Wisconsin 2016 DeCrescenzo and Mayer (2019) use survey and other data to argue that tens of thousands of electors in Wisconsin in 2016 were prevented from voting because of their beliefs about onerous voter identification requirements. Table 30 reports estimates using two eforensics model specifications, one that omits fixed effects and the other that includes county fixed effects for turnout, vote choice and eforensics-frauds magnitudes. The leader is the candidate with the most votes (Donald Trump, Republican). For both models there are strong signs of posterior MCMC multimodality: for the model with fixed effects, $D\left(\pi_{2}\right)=0$ and $M\left(\pi_{2}\right)=.181$; votes lost asymmetrically by opposition, as DeCrescenzo and Mayer (2019) argue occurred, are a likely explanation.

For the model that lacks fixed effects the posterior mean $F_{w}=24721.9$ exceeds the margin of 23089 between first-place and second-place in our data, even though approximately half of the $99.5 \%$ credible interval is smaller than the margin. For the model with fixed effects $F_{w}=16438.6$ is smaller than the margin. In both models all eforensics-frauds are incremental. Figure 25 shows that the eforensics-frauds magnitudes fixed effects are active for most counties and diverse. Milwaukee, one of the two counties on which DeCrescenzo and Mayer (2019) especially focus as having many deterred electors, has the greatest fixed effect (posterior mean) for manufactured votes. Again this is likely to be a case where eforensics represents votes lost by opposition as votes gained by the leader; allegations that voting technologies helped hijack votes for Trump have not been borne out (Mebane and Bernhard 2019).

Ohio 2004 In the 2004 election for president in Ohio studies revealed many inadequacies of election administration, including failures in voting technology provision and operations and confusing ballots (one candidate sort of remained on the ballot while being ineligible to receive votes), and there were strong allegations of biases that disadvantaged African American electors (e.g. Voting Rights Institute 2005; Mebane 2005). Long lines and other problems contributed to thousands of lost votes, although one study that relied on robust
overdispersed multinomial regression models concluded that all the various sources of losses were not enough to have changed the election outcome (Mebane and Herron 2005). The eforensics-plots shown in Figure 26 are similar to those observed in Florida 2000 (Figure 23) in that precincts with lower turnout tend to vote less favorably for the leader (George W. Bush, Republican) than do precincts with higher turnout. As does the Florida 2000 president election, the Ohio 2004 president election has relatively high efficiency (Tables 5 and 6).

Estimates of eforensics model specifications that exclude fixed effects show strong signs of posterior MCMC multimodality. Also county fixed effects are well motivated because across counties the recorded levels of turnout vary substantially due to variations in how recently counties had purged their registered voter rolls (Mebane and Herron 2005). Table 31 reports estimates for a specification that includes county fixed effects for turnout, vote choice and eforensics-frauds magnitudes. Right away we see $F_{w}=177874.2$, which is greater than the margin of 147736 between first- and second-place in our data. If eforensics-fraudulent votes are removed the leader has less than a majority of the votes: originally the leader's vote proportion is .511 , but removing eforensics-frauds produces $(2766860-177874.2) /(5411161-79378.2)=.486$. If the stolen votes are all added to the votes for the second-place candidate (John Kerry, Democrat), then that candidate's proportion is

$$
\frac{2619124+(177874.2-79378.2)}{5411161-79378.2}=.509
$$

Figure 27 shows that all the fixed effects for active incremental frauds magnitudes parameters are negative, so the incremental frauds likely include contributions from strategic elector behavior, in addition to any consequences of lost votes or of malevolent distortions of electors' intentions. In Figure 27(a,b) the value for Hamilton County sticks out as relatively high, and Hamilton is one of only two counties that have active extreme frauds (Figure 27(c,d)).

Because I believe the total of eforensics-fraudulent votes $F_{w}$ includes incremental
frauds induced by strategic elector behavior-how many I don't know-I can't say whether these eforensics estimates overturn the conclusion by Mebane and Herron (2005) that election administration failures (whether accidental or not) did not decide the election outcome. In view of $M\left(\pi_{2}\right)=.0160$, the eforensics estimates still exhibit posterior MCMC multimodality (with $D\left(\pi_{2}\right)=.998$ ).

Estimates of the eforensics model shown in Table 32 confirm the general message of Voting Rights Institute (2005) and the specific finding of Mebane (2005) that African American electors were especially adversely affected by the various deficiencies in the election process. The specification used to produce Table 32 adds to the fixed effects that were used for Table 31 an additional variable that increases as the proportion of registered voters in a precinct who are African American increases. ${ }^{27}$ The African American variable is included in the turnout, vote choice and eforensics-frauds magnitudes equations. Coefficient estimates in Table 32 show that precincts that have a higher proportion African American tend also to have lower turnout $\left(\beta_{1}<0\right)$ and a lower proportion of votes cast for the leader $\left(\gamma_{1}<0\right)$. Such precincts also tend to have larger magnitudes of eforensics-frauds ( $\rho_{M 1}>0, \rho_{S 1}>0, \delta_{M 1}>0, \delta_{S 1}>0$ ). These eforensics-frauds magnitudes coefficients most likely reflect the disparate and excessive impacts of the lost votes on African American electors: votes lost from supporters of an opposition candidate appear in eforensics estimates as eforensics-fraudulent votes for the leader. With the African American variable taken into account as in the eforensics model of Table 32, $F_{w}=117786.3$.

Signs of posterior MCMC multimodality remain apparent in the estimates reported in Table 32, indeed these are slightly stronger than they were for the model specification that omitted the African American variable: $D\left(\pi_{2}\right)=.020$ and $M\left(\pi_{2}\right)=.0181$. Perhaps, similar to the case of Argentina 2015 president round 2 (Table 28), including the additional

[^21]covariate has produced better signals about the importance of lost votes. Likely the eforensics estimates reported in Table 32 remain distorted-aliased-to an unknown extent by lost votes, even though the principal conclusions the model suggests are plausible.

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Figure 1: eforensics-plots: France 2017 National Assembly, Second Round
(a) original data

(b) district-residualized data

Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables 1.

Table 1: France 2017 National Assembly Election, Second Round, BVT Data eforensics Estimates, District Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .990 | .987 | .992 |
|  | $\pi_{2}$ | Incremental Fraud | .0102 | .00765 | .0134 |
|  | $\pi_{3}$ | Extreme Fraud | .000106 | .0000301 | .000191 |
| turnout | $\beta_{0}$ | (Intercept) | -.407 | -.422 | -.394 |
| vote choice | $\gamma_{0}$ | (Intercept) | .420 | .404 | .437 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.141 | -.264 | -.0455 |
|  | $\rho_{S 0}$ | (Intercept) | -.0598 | -.112 | -.0301 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0493 | -.133 | .0106 |
|  | $\delta_{S 0}$ | (Intercept) | .0108 | -.0466 | .0696 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.988 .^{c}$
posterior means difference $M\left(\pi_{1}\right)=.00435 ; M\left(\pi_{2}\right)=.00436 ; M\left(\pi_{3}\right)=.00000679 .{ }^{d}$
units eforensics-fraudulent: (478 incremental, 6 extreme, 68276 not fraudulent)
manufactured votes $\quad F_{t}=22471.8[20307.3,24037.5]^{e}$
incremental manufactured $\quad F_{t}=22393.4[20236.7,23961.1]^{e}$ extreme manufactured $\quad F_{t}=78.3[43.7,95.6]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=28719.6[25598.0,30806.6]^{e}$
incremental total $\quad F_{w}=28620.4[25515.9,30705.8]^{e}$
extreme total $\quad F_{w}=99.1[55.2,120.0]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). District fixed effects for turnout and vote choice are not shown. $n=68760$ bureaux units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=47289715 ; \sum_{i=1}^{n} V_{i}=18176066 ; \sum_{i=1}^{n} W_{i}=10970881 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [ $99.5 \%$ credible interval].

Table 2: 2017 Conseil Constitutionnel cases regressed on bureaux eforensics-frauds pooled by district

|  | $(\mathrm{A})$ number of cases |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
| regressor | $(\mathrm{a})^{a}$ | $(\mathrm{~b})^{b}$ | $(\mathrm{c})^{b}$ | $(\mathrm{~d})^{b}$ |
| Intercept | .00402 | -.141 | -.144 | -.145 |
| $M$ | $(.0983)$ | $[-.249,-.0313]$ | $[-.253,-.0268]$ | $[-.246,-.340]$ |
| $F_{t} /(M+1)$ | -.0000277 | - | - | - |
|  | - | - | - | - |
| $\left(F_{w}-F_{t}\right) /(M+1)$ | - | .179 | - | -.599 |
|  | - | - | - | $[-1.80,-.00713]$ |
| $n$ of districts | 572 | - | .869 | 3.55 |
| AIC $^{c}$ | 1516.1 | 1518.7 | $[.477,1.26]$ | $[.833,9.32]$ |
|  |  |  | 572 | 572 |
|  |  |  |  | 1518.1 |

(B) annulments

| regressor | $(\mathrm{e})^{b}$ | $(\mathrm{f})^{b}$ | $(\mathrm{~g})^{b}$ |
| :--- | :---: | :---: | :---: |
| Intercept | -5.76 | -5.77 | -5.79 |
|  | $[-6.55,-4.98]$ | $[-6.59,-5.00]$ | $[-6.52,-4.96]$ |
| $F_{t} /(M+1)$ | .795 | - | - |
| $\left(F_{w}-F_{t}\right) /(M+1)$ | $[.640, .974]$ | - | - |
|  | - | 3.73 | - |
| $F_{w} /(M+1)$ | - | $[2.97,4.55]$ | - |
|  | - | - | 3.73 |
| $n$ of districts | 302 | - | $[3.03,4.53]$ |
| $\mathrm{AIC}^{c}$ | 61.0 | 60.9 | 302 |
|  |  |  |  |

Note: Poisson and binomial logistic regressions of counts of French 2017 National Assembly election Conseil Constitutionnel cases and annulments (by district) on bureaux eforensics-frauds estimates using second round votes. (a-d) are from Poisson regressions for the number of cases. ( $\mathrm{e}-\mathrm{h}$ ) are from binomial logistic regressions for annulment decisions. Annulment models adjust for censoring. ${ }^{a}$ Coefficient point estimates (robust standard error) are shown. ${ }^{b}$ Coefficient Normal approximation mean and $95 \%$ confidence interval based on robust covariance matrices are shown. ${ }^{c}(\mathrm{~b}-\mathrm{h})$ AIC for models using estimates' posterior means. $M$ is the vote count difference between first and second in each district.
Upper bounds of $F_{t} /(M+1),\left(F_{w}-F_{t}\right) /(M+1)$ and $F_{w} /(M+1)$ (lower bounds are always zero): 6.12, 1.30 and 7.775 for 2017. Product-moment correlation: $\operatorname{cor}\left(F_{t} / M,\left(F_{w}-F_{t}\right) / M\right)=.981$.

Figure 2: eforensics-plots: Afghanistan 2009 President
(a) original data

(b) province-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables 3 and ??.

Figure 3: eforensics-plots: Uganda 2011 President
(a) original data

(b) province-residualized data

Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table ??.

Figure 4: eforensics-plots: Afghanistan 2009 and Uganda 2011 Partial (a) Afghanistan 2009 President

(b) Uganda 2011 President


Note: upper-right portions of Figures 2(b) and 3(b). $x$-axis is Residualized Turnout Proportion. $y$-axis is Residualized Presidential Leader Votes Proportion.

Table 3: Afghanistan 2009 President Election eforensics Estimates

| Type |  |  | $95 \% \mathrm{CI}^{a}$ |  |  |
| :--- | :---: | ---: | :---: | :---: | :---: |
|  | Parameter | Covariate | Mean | lo | up |
| mixture probabilities | $\pi_{1}$ | No Fraud | .907 | .893 | .926 |
|  | $\pi_{2}$ | Incremental Fraud | .0351 | .0201 | .0469 |
|  | $\pi_{3}$ | Extreme Fraud | .0579 | .0530 | .0621 |
| turnout | $\gamma_{0}$ | (Intercept) | -.622 | -.667 | -.579 |
| vote choice | $\beta_{0}$ | (Intercept) | -.107 | -.155 | -.0434 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | .431 | .305 | .718 |
|  | $\rho_{S 0}$ | (Intercept) | -.775 | -1.08 | -.593 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | 3.74 | 3.34 | 4.10 |
|  | $\delta_{S 0}$ | (Intercept) | 2.11 | 1.75 | 2.65 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=1$. ${ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0279 ; M\left(\pi_{2}\right)=.0238 ; M\left(\pi_{3}\right)=.00482 .{ }^{d}$
units eforensics-fraudulent: 203 incremental, 1419 extreme, 21236 not fraudulent
manufactured votes incremental manufactured extreme manufactured
total eforensics-fraudulent votes incremental total extreme total

Note: eforensics model parameter estimates (posterior means and credible intervals). $n=22858$ polling station units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=13746283 ; \sum_{i=1}^{n} V_{i}=5662758 ; \sum_{i=1}^{n} W_{i}=3093256 .{ }^{a} 95 \%$ HPD interval. ${ }^{c} \mathrm{dip}$ test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Table 4: Uganda 2011 President Election eforensics Estimates, Province Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .786 | .771 | .802 |
|  | $\pi_{2}$ | Incremental Fraud | .196 | .180 | .211 |
|  | $\pi_{3}$ | Extreme Fraud | .0182 | .0165 | .0200 |
| turnout | $\beta_{0}$ | (Intercept) | .268 | .245 | .300 |
| vote choice | $\gamma_{0}$ | (Intercept) | .806 | .770 | .832 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.389 | -.445 | -.340 |
|  | $\rho_{S 0}$ | (Intercept) | -.103 | -.176 | -.0462 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | .653 | .553 | .774 |
|  | $\delta_{S 0}$ | (Intercept) | -.432 | -.680 | -.275 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=.371 ; D\left(\pi_{2}\right)=.92 ; D\left(\pi_{3}\right)=.998 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0222 ; M\left(\pi_{2}\right)=.0224 ; M\left(\pi_{3}\right)=.000662 .{ }^{d}$
units eforensics-fraudulent: (3921 incremental, 447 extreme, 19459 not fraudulent)
manufactured votes
incremental manufactured extreme manufactured
total eforensics-fraudulent votes
incremental total extreme total
$F_{t}=225494.7[202820.2,247391.9]^{e}$
$F_{t}=158839.4[136558.9,178784.1]^{e}$
$F_{t}=66655.3[64567.6,69155.7]^{e}$
$F_{w}=312556.3[280646.5,342569.0]^{e}$
$F_{w}=226407.3[195922.6,253788.1]^{e}$
$F_{w}=86149.0[83863.6,89690.5]^{e}$

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). Province fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown. $n=23827$ polling station units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=13875338 ; \sum_{i=1}^{n} V_{i}=7928276$; $\sum_{i=1}^{n} W_{i}=5436639 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Table 5: Turnout by Leader Vote Proportion Scatterplot Gridded Entropy 1

| Event | Entropy |  |  | Sample <br> Size | Figure |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Residual. Observed Data | $\begin{gathered} \text { Normal } \\ \text { Sim. } \\ \text { Data } \end{gathered}$ | Residual. <br> Data <br> Efficiency |  |  |
| Afghanistan 2009 President | 8.06 | 9.96 | . 9611 | 22858 | 2(b) |
| Argentina 2015 President, Round 1 | 9.04 | 10.7 | . 9732 | 92211 | 20(b) |
| Argentina 2015 President, Round 2 | 9.22 | 10.9 | . 9759 | 92632 | 21(b) |
| Bangladesh 2001 | 8.88 | 10.1 | . 9935 | 29499 | 15(b) |
| California 2006 Gov., Partisan | 8.70 | 9.94 | . 9932 | 22820 | ?? (b) |
| California 2006 Gov., County | 8.83 | 9.95 | . 9940 | 22820 | ?? (b) |
| California 2008 Pres., Partisan | 8.31 | 9.89 | . 9903 | 21420 | ?? (b) |
| California 2008 Pres., County | 8.32 | 9.93 | . 9920 | 22691 | ?? (a) |
| Canada 2011 Legislature | 9.24 | 10.8 | . 9708 | 70303 | 5(b) |
| Florida 2000 President | 7.61 | 8.66 | . 9986 | 5941 | 23(b) |
| France 2017 N Assembly, Round 1 | 9.28 | 10.7 | . 9822 | 69240 | ??(b) |
| France 2017 N Assembly, Round 2 | 9.30 | 10.6 | . 9842 | 68760 | 1(b) |
| Germany 2005 Erststimmen | 9.51 | 10.9 | . 9832 | 88680 | ??(b) |
| Germany 2005 Zweitstimmen | 9.57 | 10.9 | . 9823 | 88680 | ?? (d) |
| Germany 2021 Erststimmen | 9.77 | 11.0 | . 9902 | 94248 | ?? (b) |
| Germany 2021 Zweitstimmen | 9.73 | 10.9 | . 9849 | 94248 | ?? (d) |
| Kenya 2017 President | 8.80 | 10.3 | . 9760 | 40818 | ??(b) |
| Kenya 2022 President | 9.08 | 10.5 | . 9859 | 46214 | ??(b) |
| Mexico 2006 Deputies | 9.72 | 11.2 | . 9815 | 130448 | 14(b) |
| Mexico 2006 President | 9.97 | 11.2 | . 9867 | 130768 | 6(b) |
| Ohio 2004 President | 8.37 | 9.27 | . 9958 | 11123 | 26(b) |
| Ohio 2006 U.S. Senate | 7.50 | 9.26 | . 9971 | 11123 | ??(b) |

Note: bivariate (residualized turnout by residualized leader vote proportion) entropy measures. To compute entropy observations are mapped into grids of $1000 \times 1000$ cells. ${ }^{a}$ Normal simulation data are generated to have the same covariance matrix as does the corresponding residualized observed data.

Table 6: Turnout by Leader Vote Proportion Scatterplot Gridded Entropy 2

| Event | Entropy |  |  | Sample <br> Size | Figure |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Residual. Observed Data | Normal Sim. Data ${ }^{a}$ | Residual. <br> Data Efficiency |  |  |
| Peru 2021 President Round 1 | 8.93 | 10.9 | . 9579 | 83366 | ?? (b) |
| Peru 2021 President Round 2 | 9.10 | 11.0 | . 9627 | 83366 | ?? (b) |
| Philippines 2022 President | 9.34 | 11.0 | . 9740 | 105649 | ??(b) |
| Russia 2011 Duma | 10.3 | 11.1 | . 9896 | 95166 | 8(b) |
| Russia 2012 President | 10.1 | 11.1 | . 9859 | 95413 | ??(b) |
| Russia 2020 Referendum | 10.2 | 11.0 | . 9866 | 96239 | ?? (b) |
| South Africa 2014 National | 8.76 | 9.93 | . 9970 | 22260 | ??(b) |
| Turkey 1999 Legislature | 9.97 | 11.3 | . 9769 | 208474 | ?? (b) |
| Turkey 2011 Legislature | 10.1 | 11.3 | . 9775 | 199555 | ??(b) |
| Turkey 2015 June | 9.96 | 11.1 | . 9780 | 173850 | ??(a) |
| Turkey 2015 November | 9.99 | 11.1 | . 9775 | 174619 | ?? (b) |
| Turkey 2017 Referendum | 9.92 | 11.1 | . 9768 | 171352 | ?? (b) |
| Türkiye 2023 Legislature | 9.73 | 11.1 | . 9681 | 191875 | ?? (b) |
| Türkiye 2023 President, Round 1 | 9.89 | 11.1 | . 9729 | 191863 | ?? (b) |
| Uganda 2006 President | 8.64 | 9.81 | . 9956 | 19750 | ??(b) |
| Uganda 2011 President | 8.84 | 9.98 | . 9947 | 23827 | 3(b) |
| Washington 2008 Init. 1000 | 7.36 | 8.51 | . 9924 | 5180 | ?? (b) |
| Washington 2008 Init. 1029 | 6.97 | 8.52 | . 9899 | 5180 | ??(d) |

Note: bivariate (residualized turnout by residualized leader vote proportion) entropy measures. To compute entropy observations are mapped into grids of $1000 \times 1000$ cells. ${ }^{a}$ Normal simulation data are generated to have the same covariance matrix as does the corresponding residualized observed data.

Figure 5: eforensics-plots: Canada 2011
(a) original data
(b) district- and mobile/SVRresidualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table 15.

Figure 6: eforensics-plots: Mexico 2006 President
(a) original data

(b) estado- and casilla-type-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table ??.

Figure 7: eforensics-plots: Canada 2011 and Mexico 2006 Partial
(a) Canada 2011 Federal

(b) Mexico 2006 President


Note: upper-right portions of Figures 5(b) and 6(b). $x$-axis is Residualized Turnout Proportion. $y$-axis is Residualized Presidential Leader Votes Proportion.

Figure 8: Residualized eforensics-plots: Russia 2011 Duma
(a) original data

(b) region-residualized data

Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables ??, ?? and (Moscow only) 22 and 23.

Table 7: California 2008 Ballot Propositions Proportions and Tests

| proposition | total votes | total vote proportion | $\begin{gathered} \text { YES } \\ \text { votes } \end{gathered}$ | $\begin{gathered} \text { NO } \\ \text { votes } \end{gathered}$ | multimodality tests |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $p$-value ${ }^{a}$ | $\operatorname{max~M~}{ }^{\text {b }}$ |
| 1A | 12653092 | . 92 | . 49 | . 44 | 0 | . 0225 |
| 2 | 12891960 | . 94 | . 60 | . 34 | . 0175 | . 0104 |
| 3 | 12595906 | . 92 | . 51 | . 41 | 0 | . 0252 |
| 4 | 12905377 | . 94 | . 45 | . 49 | 0 | . 0263 |
| 5 | 12678632 | . 93 | . 38 | . 55 | . 939 | . 0107 |
| 6 | 12341371 | . 90 | . 28 | . 62 | 0 | . 131 |
| 7 | 12614558 | . 92 | . 33 | . 59 | 0 | . 0648 |
| 8 | 13357973 | . 98 | . 51 | . 47 | 0 | . 0151 |
| 9 | 12368772 | . 90 | . 49 | . 42 | . 0005 | . 000925 |
| 10 | 12519816 | . 91 | . 37 | . 54 | 0 | . 229 |
| 11 | 11951347 | . 87 | . 44 | . 43 | 0 | . 0209 |
| 12 | 12245801 | . 89 | . 57 | . 33 | 0 | . 00367 |

Note: ${ }^{a}$ Minimum $p$-value from dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains for $\pi_{1}, \pi_{2}$ or $\pi_{3} .{ }^{b}$ Maximum difference across $\pi_{1}, \pi_{2}$ and $\pi_{3}$ between largest and smallest chain-specific posterior means.
Proposition descriptions (results): 1A, high-speed rail bonds (Y); 2, minimum space requirements for calves, pigs and hens (Y); 3, children's hospitals bonds (Y); 4, minor abortion parental notification (N); 5, drug crime policy, sentencing and rehabilitation (N); 6 , gang-related criminal laws, law enforcement funding, and parole agent caseloads (N); 7, renewable portfolio standards (N); 8, defines marriage as between one man and one woman, reversing legal same-sex marriage in California (Y); 9, rights of crime victims (Y); 10 , alternative fuel project bonds (N); 11, legislative redistricting power (N); 12, bonds to provide loans to veterans to purchase homes or farms (Y). Descriptions edited from https://ballotpedia.org/California_2008_ballot_propositions.

Table 8: California 2008 Ballot Propositions 1A-5 eforensics-fraudulent Votes
(a) Proposition 1A:
units eforensics-fraudulent: (3 incremental, 8 extreme, 22679 not fraudulent)

| manufactured votes | $F_{t}=744.0[381.3,1478.7]^{a}$ |
| :---: | :---: |
| incremental manufactured | $F_{t}=128.3[0.0,243.8]^{a}$ |
| extreme manufactured | $F_{t}=615.7[249.3,1407.0]^{a}$ |
| total eforensics-fraudulent votes | $F_{w}=1415.0[768.6,2556.2]^{a}$ |
| incremental total | $F_{w}=235.9[0.0,470.6]^{a}$ |
| extreme total | $F_{w}=1179.1[535.2,2357.3]^{a}$ |

(b) Proposition 2:
units eforensics-fraudulent: (4 incremental, 8 extreme, 22677 not fraudulent)

| manufactured votes | $F_{t}=566.1[250.8,968.3]^{a}$ |
| :---: | :---: |
| incremental manufactured | $F_{t}=189.9[0.0,426.3]^{a}$ |
| extreme manufactured | $F_{t}=376.2[217.1,552.3]^{a}$ |
| total eforensics-fraudulent votes | $F_{w}=1341.7[731.4,2128.5]^{a}$ |
| incremental total | $F_{w}=256.2[0.0,569.2]^{a}$ |
| extreme total | $F_{w}=1085.5[652.7,1572.2]^{a}$ |

(c) Proposition 3:
units eforensics-fraudulent: (2 incremental, 0 extreme, 22689 not fraudulent)
manufactured votes $\quad F_{t}=64.1[0.0,164.7]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=157.1[0.0,443.4]^{a}$
(d) Proposition 4:
units eforensics-fraudulent: (2012 incremental, 12 extreme, 20666 not fraudulent)
manufactured votes $\quad F_{t}=99814.4[87429.4,125829.6]^{a}$
incremental manufactured $\quad F_{t}=98437.9[86072.3,123705.3]^{a}$ extreme manufactured $\quad F_{t}=1376.5[904.1,2163.0]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=266540.5[245813.8,283159.5]^{a}$ incremental total $\quad F_{w}=263205.1[242311.1,280121.3]^{a}$ extreme total $\quad F_{w}=3335.4[2506.8,4184.0]^{a}$
(e) Proposition 5:
units eforensics-fraudulent: (1247 incremental, 6 extreme, 21435 not fraudulent)

$$
\begin{array}{cc}
\text { manufactured votes } & F_{t}=37184.3[30792.6,41689.9]^{a} \\
\text { incremental manufactured } & F_{t}=36918.7[30522.0,41427.1]^{a} \\
\text { extreme manufactured } & F_{t}=265.6[164.5,355.7]^{a} \\
\text { total eforensics-fraudulent votes } & F_{w}=57579.5[48242.2,63007.2]^{a} \\
\text { incremental total } & F_{w}=56914.4[47678.0,62364.2]^{a} \\
\text { extreme total } & F_{w}=665.1[410.8,934.3]^{a}
\end{array}
$$

Note: ${ }^{a}$ posterior mean [99.5\% credible interval].

Table 9: California 2008 Ballot Propositions 6-12 eforensics-fraudulent Votes
(a) Proposition 6:
units eforensics-fraudulent: (2215 incremental, 7 extreme, 20468 not fraudulent)

| manufactured votes | $F_{t}=78140.8[33058.9,99263.9]^{a}$ |
| :---: | :---: |
| incremental manufactured | $F_{t}=77789.0[32848.2,98844.0]^{a}$ |
| extreme manufactured | $F_{t}=351.9[95.9,456.5]^{a}$ |
| total eforensics-fraudulent votes | $F_{w}=117009.2[48898.9,145697.8]^{a}$ |
| incremental total | $F_{w}=116033.1[48606.1,144541.3]^{a}$ |
| extreme total | $F_{w}=976.0[171.0,1385.5]^{a}$ |

(b) Proposition 7:
units eforensics-fraudulent: (10541 incremental, 3 extreme, 12144 not fraudulent) manufactured votes $\quad F_{t}=434512.8[376824.6,480583.7]^{a}$
incremental manufactured $\quad F_{t}=434260.8[376570.9,480300.8]^{a}$
extreme manufactured $\quad F_{t}=252.0[155.0,293.9]^{a}$
total eforensics-fraudulent votes
incremental total
$F_{w}=681066.7[593296.8,741868.1]^{a}$
$\quad F_{w}=680545.1[592737.1,741285.5]^{a}$ extreme total $\quad F_{w}=521.6[314.9,600.1]^{a}$
(c) Proposition 8:
units eforensics-fraudulent: (33 incremental, 4 extreme, 22652 not fraudulent)
manufactured votes incremental manufactured extreme manufactured
total eforensics-fraudulent votes incremental total extreme total

$$
F_{t}=1613.2[1034.6,2095.8]^{a}
$$

$$
F_{t}=1443.4[976.5,1822.3]^{a}
$$

$$
F_{t}=169.9[48.3,293.6]^{a}
$$

$$
F_{w}=2951.2[1941.4,3874.3]^{a}
$$

$$
F_{w}=2576.2[1766.3,3300.9]^{a}
$$

$$
F_{w}=375.0[144.5,607.0]^{a}
$$

(d) Proposition 9:
units eforensics-fraudulent: (0 incremental, 0 extreme, 22689 not fraudulent)
(e) Proposition 10:
units eforensics-fraudulent: (6991 incremental, 0 extreme, 15699 not fraudulent)

| manufactured votes | $F_{t}=336897.4[207955.8,408075.6]^{a}$ |
| :--- | :--- |
|  | $F^{a}=497603.5[293969.0,594491.7]^{a}$ |

total eforensics-fraudulent votes $\quad F_{w}=497603.5[293969.0,594491.7]^{a}$
(f) Proposition 11:
units eforensics-fraudulent: (832 incremental, 0 extreme, 21858 not fraudulent)
manufactured votes $\quad F_{t}=27254.2[17434.0,36843.8]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=45895.1[33894.3,55132.5]^{a}$
(g) Proposition 12:
units eforensics-fraudulent: (4 incremental, 0 extreme, 22686 not fraudulent) manufactured votes $\quad F_{t}=118.3[32.4,249.1]^{a}$ total eforensics-fraudulent votes $\quad F_{w}=308.9[79.1,772.0]^{a}$

Note: ${ }^{a}$ posterior mean [99.5\% credible interval].

Table 10: Numbers of Constitutional Amendments, Ballot Propositions, Etc. by Election Year

| $c$ | election | num | election | num | election |
| :---: | ---: | :--- | ---: | :--- | ---: | num

Figure 9: Ballot Proposition Proportions and MCMC Multimodality


Note: maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means ( $y$-axis) by (a) the proportion voting either yes or no on a statewide ballot propostion or (b) maximum of proportion voting yes or proportion voting no ( $x$-axis). $n=75$ (observation count by state: CA 45; AK 2; FL 12; GA 10; VA 3; WA 3).

Table 11: MCMC Multimodality ${ }^{a}$ Conditioned on Ballot Proposition Voting Proportions

| covariate | $(1)$ | $(2)$ | $(3)$ |
| :--- | ---: | ---: | ---: |
| (Intercept) | -.00602 | -.0861 | 1.65 |
|  | $(.183)$ | $(.0464)$ | $(1.34)$ |
| proportion voting | .0480 | - | -2.77 |
|  | $(.197)$ | - | $(2.22)$ |
| max proportion Yes or No | - | .199 | -1.86 |
|  | - | $(.0777)$ | $(1.43)$ |
| prop. voting $\times \max Y$ or N | - | - | 3.18 |
|  | - | - | $(2.38)$ |
| RMSE | .0673 | .0649 | .0651 |
| Adjusted $R^{2}$ | -.01311 | .0588 | .0524 |

Note: regression model of $M\left(\pi_{2}\right)$ coefficient estimates (robust standard error). $n=75$ constitutional amendments, initiatives, propositions and referenda (observation count by state: CA 45; AK 2; FL 12; GA 10; VA 3; WA 3). ${ }^{a} M\left(\pi_{2}\right)$ : maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means.

Figure 10: MCMC Multimodality and eforensics-frauds


Note: eforensics-fraudulent votes as a proportion of the number of voters $\left(F_{w} / N\right)$ ( $y$-axis) by maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means ( $x$-axis). $n=75$ (observation count by state: CA 45; AK 2; FL 12; GA 10; VA 3; WA 3).

Figure 11: MCMC Multimodality and eforensics-fraud Components B


Note: components of eforensics-fraudulent votes as a proportion of the number of voters ( $y$-axis) by maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means ( $x$-axis). $n=75$ (observation count by state: CA 45; AK 2; FL 12; GA 10; VA 3; WA 3).

Table 12: MCMC Multimodality and eforensics-frauds Among Ballot Propositions

|  | Estimates $^{a}$ |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
| covariate | incremental |  | extreme |  |
| manuf. | stolen | manuf. | stolen |  |
| (Intercept) | -6.32 | -6.28 | -8.47 | -8.76 |
|  | $(.0489)$ | $(.0581)$ | $(.0705)$ | $(.0514)$ |
| $M\left(\pi_{2}\right)^{b}$ | 8.21 | 2.84 | 17.2 | 9.25 |
|  | $(.295)$ | $(.382)$ | $(.428)$ | $(.314)$ |
| $D\left(\pi_{2}\right)^{c}$ | .749 | .666 | .512 | .978 |
| $D\left(\pi_{2}\right) \times M\left(\pi_{2}\right)$ | $(.0951)$ | $(.102)$ | $(.136)$ | $(.118)$ |
|  | 63.4 | 46.5 | 54.4 | -121. |
| AIC | $(2.27)$ | $(3.33)$ | $(7.92)$ | $(8.21)$ |
| 66226294 |  |  |  |  |

Note: outcomes are estimated numbers of eforensics-fraudulent votes by type for each ballot proposition (summing all precincts). $n=75$ constitutional amendments, initiatives, propositions and referenda (observation count by state: CA 45; AK 2; FL 12; GA 10; VA 3; WA 3). ${ }^{a}$ MNL regression model coefficient estimates (robust standard error) by eforensics-fraud type: reference category is "no frauds." ${ }^{b}$ Maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means. ${ }^{c}$ All-chains dip test $p$-value for $\pi_{2}$.

Figure 12: eforensics-plots: Germany 2021 Bundestag Erststimmen
(a) original data
(b) Wahlkreis- and Wahlbezirk-typeresidualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table 13.

Table 13: Germany 2021 Election Erststimmen eforensics Estimates, Wahlkreis Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .994 | .991 | .999 |
|  | $\pi_{2}$ | Incremental Fraud | .00589 | .000892 | .00934 |
| turnout | $\pi_{3}$ | Extreme Fraud | $1.05 \mathrm{e}-05$ | $9.71 \mathrm{e}-10$ | $3.26 \mathrm{e}-05$ |
|  | $\beta_{0}$ | (Intercept) | -.586 | -1.28 | .297 |
| vote choice | $\beta_{1}$ | Briefwahl | -.808 | -1.15 | -.393 |
|  | $\beta_{2}$ | special | .604 | -.660 | 2.45 |
| incremental frauds | $\gamma_{0}$ | (Intercept) | -.728 | -.769 | -.692 |
|  | $\gamma_{1}$ | Briefwahl | .0750 | .0662 | .0817 |
| extreme frauds | $\gamma_{2}$ | special | -.000809 | -.101 | .0786 |
|  | $\rho_{S 0}$ | (Intercept) | -.0726 | -.123 | .00427 |
|  | $\delta_{M 0}$ | (Intercept) | -.0444 | -.0795 | .00554 |
|  | $\delta_{S 0}$ | (Intercept) | .00630 | .00154 | .0124 |
|  | (Intercept) | -.00739 | -.0191 | -.00162 |  |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=1$. ${ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.00787 ; M\left(\pi_{2}\right)=.00787 ; M\left(\pi_{3}\right)=1.22 \mathrm{e}-06 .^{c}$
units eforensics-fraudulent: (194 incremental, 0 extreme, 94054 not fraudulent)
manufactured votes $\quad F_{t}=10130.7[6289.0,13786.5]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=15031.5[7449.6,19100.4]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). Wahlkreis fixed effects for turnout and vote choice are not shown. $n=94248$ polling station and Briefwahlbezirke units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=61180662 ; \sum_{i=1}^{n} V_{i}=24389256 ; \sum_{i=1}^{n} W_{i}=15463135$. ${ }^{a}$ $95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean $[99.5 \%$ credible interval].

Figure 13: eforensics-plots: Germany 2005 Bundestag Erststimmen
(a) original data
(b) Wahlkreis- and Wahlbezirk-typeresidualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table 14

Table 14: Germany 2005 Election Erststimmen eforensics Estimates, Wahlkreis Fixed Effects

| Type | Parameter | Covariate | Mean | lo $^{a}$ | up $^{b}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .937 | .923 | .946 |
|  | $\pi_{2}$ | Incremental Fraud | .0632 | .0541 | .0772 |
| turnout | $\pi_{3}$ | Extreme Fraud | $7.11 \mathrm{e}-05$ | $9.31 \mathrm{e}-06$ | .000142 |
|  | $\beta_{0}$ | (Intercept) | .435 | -.0343 | 1.20 |
|  | $\beta_{1}$ | Briefwahl | -1.19 | -1.98 | -.683 |
| vote choice | $\beta_{2}$ | special | .553 | -1.47 | 2.08 |
|  | $\gamma_{0}$ | (Intercept) | -.203 | -.227 | -.162 |
| incremental frauds | $\gamma_{1}$ | Briefwahl | .0724 | .0566 | .0867 |
|  | $\gamma_{2}$ | special | -.0281 | -.106 | .0278 |
| extreme frauds | $\rho_{M 0}$ | (Intercept) | -.168 | -.202 | -.139 |
|  | $\rho_{S 0}$ | (Intercept) | -.210 | -.251 | -.187 |
|  | $\delta_{M 0}$ | (Intercept) | -.00981 | -.0250 | .00226 |
|  | $\delta_{S 0}$ | (Intercept) | -.00987 | -.0309 | .0184 |

posterior multimodality diagnostics:
Erst.: all-chains dip test p-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.987 .{ }^{c}$
posterior means difference $\quad M\left(\pi_{1}\right)=.0180 ; M\left(\pi_{2}\right)=.0179 ; M\left(\pi_{3}\right)=.0000537 .{ }^{d}$
units eforensics-fraudulent: (3750 incremental, 1 extreme, 84929 not fraudulent)
manufactured votes
incremental manufactured
$F_{t}=217334.1[163411.2,254916.2]^{e}$
$F_{t}=217287.5[163392.1,254831.2]^{e}$
extreme manufactured
$F_{t}=46.63353[0.00000,93.06911]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=360137.2$ [344639.3, 378234.7] ${ }^{e}$
incremental total
$F_{w}=360069.4[344545.9,378234.7]^{e}$
extreme total
$F_{w}=67.84988[0.00000,104.83590]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). $n=88680$ polling station and Briefwahlbezirk units. Wahlkreis fixed effects for turnout and vote choice are not shown. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=61864986 ; \sum_{i=1}^{n} V_{i}=38332241 ; \sum_{i=1}^{n} W_{i}=22062755 .{ }^{a}$ $95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean $[99.5 \%$ credible interval].

Table 15: Canada 2011 Legislative Election eforensics Estimates, District Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | up $^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .983 | .978 | .987 |
|  | $\pi_{2}$ | Incremental Fraud | .0160 | .0121 | .0209 |
| turnout | $\pi_{3}$ | Extreme Fraud | .00126 | .000809 | .00179 |
|  | $\beta_{0}$ | (Intercept) | .0775 | .0312 | .112 |
|  | $\beta_{1}$ | Adjusted Electors | 4.86 | 4.27 | 5.50 |
|  | $\beta_{2}$ | Mobile | .179 | .0740 | .264 |
|  | $\beta_{3}$ | SVR Group 1 | -.0125 | -.0689 | .0798 |
|  | $\beta_{4}$ | SVR Group 2 | .322 | .184 | .443 |
| vote choice | $\gamma_{0}$ | (Intercept) | .00233 | -.0112 | .0165 |
|  | $\gamma_{1}$ | Adjusted Electors | -.0381 | -.0722 | -.00996 |
|  | $\gamma_{2}$ | Mobile | -.231 | -.264 | -.196 |
|  | $\gamma_{3}$ | SVR Group 1 | -.335 | -.405 | -.263 |
| incremental frauds | $\gamma_{4}$ | SVR Group 2 | -.114 | -.178 | -.0495 |
|  | $\rho_{M 0}$ | (Intercept) | .0803 | .00754 | .136 |
| extreme frauds | $\rho_{S 0}$ | (Intercept) | -.484 | -.573 | -.324 |
|  | $\delta_{M 0}$ | (Intercept) | .0934 | .0259 | .228 |
|  | $\delta_{S 0}$ | (Intercept) | -1.36 | -1.77 | -1.05 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=.0005 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.983 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.00584 ; M\left(\pi_{2}\right)=.00558 ; M\left(\pi_{3}\right)=.000507 .{ }^{d}$
units eforensics-fraudulent: (534 incremental, 83 extreme, 69686 not fraudulent)
manufactured votes
incremental manufactured $\quad F_{t}=17786.7[15105.520384 .7]^{e}$
extreme manufactured $\quad F_{t}=532.7[419.6653 .8]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=33238.6[28899.6,36959.1]^{e}$
incremental total $\quad F_{w}=26220.2[23021.2,30100.7]^{e}$ extreme total $\quad F_{w}=7018.4[5864.6,8278.1]^{e}$

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). District fixed effects for turnout and vote choice are not shown. $n=70303$ poll units. Electors, valid votes and votes for the leader:
$\sum_{i=1}^{n} N_{i}=26120334 ; \sum_{i=1}^{n} V_{i}=14480796 ; \sum_{i=1}^{n} W_{i}=7307339$. $^{a} 95 \%$ HPD lower bound.
${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Table 16: Canada 2004-2015 Legislative Election Poll eforensics-fraud Type Frequencies eforensics-frauds type

| kind of poll | no frauds | incremental | extreme | $n$ |
| :--- | ---: | ---: | ---: | ---: |
| Regular Residential | .993 | .006 | .001 | 332512 |
| Mobile | .953 | .043 | .004 | 7407 |
| Special Voting Rules 1 | .969 | .029 | .002 | 1561 |
| Special Voting Rules 2 | .952 | .037 | .012 | 1561 |

Note: Proportion of each kind of poll that is classified as having each type of eforensics-fraud, summing over estimates for 2004, 2006, 2008, 2011 and 2015.
eforensics-frauds estimates come from the models reported in Tables 15, ??, ??, ?? and ??. The $n$ column reports the number of polls of each kind.

Figure 14: eforensics-plots: Mexico 2006 Deputies
(a) original data

(b) district-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table 17.

Table 17: Mexico 2006 Deputies Election eforensics Estimates, District Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .964 | .962 | .965 |
|  | $\pi_{2}$ | Incremental Fraud | .0363 | .0350 | .0375 |
|  | $\pi_{3}$ | Extreme Fraud | .000153 | $8.01 \mathrm{e}-05$ | .000237 |
| turnout | $\beta_{0}$ | (Intercept) | .258 | .233 | .275 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.273 | -.283 | -.257 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.125 | -.150 | -.106 |
|  | $\rho_{S 0}$ | (Intercept) | -.912 | -1.02 | -.806 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | .0210 | -.0128 | .0415 |
|  | $\delta_{S 0}$ | (Intercept) | -.00761 | -.0315 | .0428 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=1 ; D\left(\pi_{2}\right)=1 ; D\left(\pi_{3}\right)=1$. ${ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.000829 ; M\left(\pi_{2}\right)=.000870 ; M\left(\pi_{3}\right)=.0000512^{d}$
units eforensics-fraudulent: (2633 incremental, 19 extreme, 127796 not fraudulent)
manufactured votes $\quad F_{t}=129840.2[120517.5,136804.5]^{a}$
incremental manufactured $\quad F_{t}=127756.4[118283.2,134660.3]^{a}$
extreme manufactured

$$
F_{t}=2083.8[1887.6,2246.4]^{a}
$$

total eforensics-fraudulent votes
incremental total
extreme total
$F_{w}=186060.1[173918.6,195023.2]^{a}$
$F_{w}=182738.7[170328.8,191852.0]^{a}$
$F_{w}=3321.4[3029.4,3617.2]^{a}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). District fixed effects for turnout and vote choice are not shown. $n=130448$ casillas (Mayoria Relativa votes). Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=64220905 ; \sum_{i=1}^{n} V_{i}=25800523 ; \sum_{i=1}^{n} W_{i}=11914080 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means.

Figure 15: eforensics-plots: Bangladesh 2001
(a) original data

(b) district-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Table 18.

Table 18: Bangladesh 2001 Elections eforensics Estimates, District Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .756 | .747 | .764 |
|  | $\pi_{2}$ | Incremental Fraud | .235 | .227 | .243 |
|  | $\pi_{3}$ | Extreme Fraud | .00940 | .00823 | .0106 |
| turnout | $\beta_{0}$ | (Intercept) | 1.12 | 1.09 | 1.13 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.117 | -.144 | -.0967 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.330 | -.365 | -.286 |
|  | $\rho_{S 0}$ | (Intercept) | -.651 | -.712 | -.568 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.617 | -.742 | -.513 |
|  | $\delta_{S 0}$ | (Intercept) | -.203 | -.242 | -.165 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=.999 ; D\left(\pi_{2}\right)=1 ; D\left(\pi_{3}\right)=.988 .^{c}$
posterior means difference $M\left(\pi_{1}\right)=.00982 ; M\left(\pi_{2}\right)=.00974 ; M\left(\pi_{3}\right)=.000643 .{ }^{d}$
units eforensics-fraudulent: (3558 incremental, 307 extreme, 25634 not fraudulent)
manufactured votes $\quad F_{t}=550673.8$ [529468.6, 578704.0] ${ }^{e}$
incremental manufactured $\quad F_{t}=451158.9$ [429183.3, 477440.5]
extreme manufactured
$F_{t}=99514.9[94674.3,103825.4]^{e}$
total eforensics-fraudulent votes
incremental total extreme total

$$
F_{w}=1387497.6[1348242.3,1428884.8]^{e}
$$

$$
F_{w}=1113899.5[1083778.7,1152018.5]^{e}
$$

$$
F_{w}=273598.0[262312.4,2852334.0]^{e}
$$

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). District fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown (see Figure 16). $n=29499$ polling station units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=73697102$; $\sum_{i=1}^{n} V_{i}=55230753 ; \sum_{i=1}^{n} W_{i}=28967523 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Figure 16: Bangladesh 2001: eforensics-frauds Magnitude Fixed Effect Parameters


Note: active fixed effects parameters (posterior means and $95 \%$ HPD intervals) for frauds magnitude ( $\rho_{M j}, \rho_{S j}, \delta_{M j}, \delta_{S j}$ ) parameters in the eforensics model reported in Table 18.. Districts with eforensics-frauds are (except for position 0, district numbers correspond to position numbers): (incremental) 0 1, 2-29, 31-114, 117-136, 138-150, 152-156, 158-167, 169-188, 191-260, 263-273, 275-279, 280-300; (extreme) 1, 5, 7, 9, 11, 18, 37, 42-44, 48, 50, $61,62,69,80,83,86,89,91-95,97,99,103,105-108,110,113,114,117-119,121,127,128$, 141-146, 154, 163, 169, 179, 182, 193, 196, 197, 199, 200, 201, 203, 209, 213-220, 222-225, $229,230,232,241-244,246-249,253,258,259,261,266-268,281,282,284,291,297,298$.

Table 19: Bangladesh 2001: Districts where eforensics-fraudulent Votes Exceed the Margin between First and Second

| district | electors | votes <br> cast | first <br> place | second <br> place | margin | eforensics- <br> frauds |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 114 | 141226 | 98890 | 45932 | 45903 | 29 | 1092.7 |
| 121 | 150629 | 116161 | 52415 | 50122 | 2293 | 4050.0 |
| 129 | 194911 | 150723 | 55702 | 55435 | 267 | 1351.1 |
| 169 | 195329 | 146272 | 54692 | 54073 | 619 | 4261.6 |
| 200 | 204236 | 169236 | 78721 | 77620 | 1101 | 1892.9 |
| 203 | 151901 | 125966 | 58947 | 58388 | 559 | 3992.9 |
| 210 | 154163 | 119361 | 55115 | 54570 | 545 | 2720.2 |
| 228 | 269109 | 183213 | 59656 | 58985 | 671 | 1235.9 |
| 253 | 133338 | 97420 | 37089 | 36724 | 365 | 792.3 |

Note: eforensics-frauds are sums of each district's polling station posterior mean estimates $F_{w i}$ from the eforensics model reported in Table 18.

Figure 17: MCMC Multimodality Measures and Tests
(a) Bangladesh 1991, 1996, $2001^{a}$

(c) Germany 2002, 2005, 2009, $2021^{c}$

(b) Canada 2004, 06, 08, 11, $2015^{b}$

(d) Mexico 2006, 2009, $2012^{d}$


Note: maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means ( $M\left(\pi_{2}\right), x$-axis) by $p$-value for posterior multimodality tests for $\pi_{2}\left(D\left(\pi_{2}\right)\right.$, $y$-axis), using nonpooled (district-specific) eforensics estimates. Lines show OLS regressions of $p$-values on mean differences.
${ }^{a}$ Bangladesh $1991(n=164)$, $1996(n=295)$, $2001(n=299) .{ }^{b}$ Canada $2004(n=308)$, $2006(n=308)$, $2008(n=304)$, $2011(n=303)$, $2015(n=261) .{ }^{c}$ Germany Erststimmen 2002, 2005, 2009, 2021 (each year $n=299$ ). ${ }^{d}$ Mexico Deputies Mayoría Relativa 2006, 2009, 2012 (each year $n=300$ ).

Table 20: MCMC Multimodality and eforensics-frauds Occurrences in Several Districted Legislative Elections

| variable | $(\mathrm{a})$ |  | $(\mathrm{b})$ |  |
| :--- | ---: | ---: | ---: | :---: |
| coef. | SE | coef. | SE |  |
| (Intercept) $_{M\left(\pi_{2}\right)^{a}}$ | -3.91 | .171 | -4.01 | .171 |
| $D\left(\pi_{2}\right)^{b}$ | 3.38 | .406 | 2.29 | .396 |
| $M\left(\pi_{2}\right) \times D\left(\pi_{2}\right)$ | -.477 | .130 | -1.43 | .124 |
| fixed effects: | - |  | 20.9 | 1.33 |
| Bangladesh 1996 | .610 | .151 | .255 | .172 |
| Bangladesh 2001 | .669 | .151 | .207 | .181 |
| Canada 2004 | .438 | .198 | .750 | .212 |
| Canada 2006 | .448 | .201 | .852 | .209 |
| Canada 2008 | .264 | .194 | .693 | .209 |
| Canada 2011 | .274 | .179 | .750 | .191 |
| Canada 2015 | .311 | .219 | .806 | .233 |
| Germany 2002 | .878 | .159 | 1.27 | .166 |
| Germany 2005 | .557 | .170 | .952 | .176 |
| Germany 2009 | -.0599 | .189 | .475 | .187 |
| Germany 2021 | -1.68 | .348 | -.846 | .349 |
| Mexico 2006 | .347 | .174 | .892 | .189 |
| Mexico 2009 | -.290 | .172 | .315 | .190 |
| Mexico 2012 | -.181 | .184 | .409 | .201 |
| AIC | 94195 | 83820 |  |  |

Note: outcomes are counts of polling stations in each district that are classified as either having or lacking eforensics-frauds using nonpooled (district-specific) eforensics estimates: reference category is "no frauds;" incremental and extreme frauds counts are combined. Binomial regression model coefficient estimates with robust standard errors. $n=4718$ legislative districts. For the fixed effects Bangladesh 1991 is the reference category. ${ }^{a}$ Maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means. ${ }^{b}$ All-chains dip test $p$-value for $\pi_{2}$.

Table 21: MCMC Multimodality and eforensics-fraudulent Votes in Several Districted Legislative Elections

| variable | pooled $F_{w i}$ estimates <br> (a) <br> (b) |  |  |  | nonpooled $F_{w i}$ estimates <br> (c) <br> (d) |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | coef. | SE | coef. | SE | coef. | SE | coef. | SE |
| (Intercept) | -4.86 | . 0741 | -4.38 | . 0716 | -5.19 | . 0895 | -4.81 | . 0867 |
| $M\left(\pi_{2}\right)^{a}$ | 2.93 | . 0931 | 1.13 | . 0861 | 3.56 | . 0861 | 1.87 | . 0698 |
| $D\left(\pi_{2}\right)^{b}$ | -. 0377 | . 0321 | -. 988 | . 0308 | -. 0297 | . 0356 | -1.06 | . 0312 |
| $M\left(\pi_{2}\right) \times D\left(\pi_{2}\right)$ |  |  | 10.8 | . 252 | - |  | 13.6 | . 0906 |
| fixed effects: |  |  |  |  |  |  |  |  |
| Bangladesh 1996 | -. 147 | . 0807 | -. 368 | . 0810 | . 366 | . 0902 | . 0940 | . 0894 |
| Bangladesh 2001 | 1.03 | . 0704 | . 772 | . 0708 | . 335 | . 0891 | . 00257 | . 0860 |
| Canada 2004 | -2.27 | . 118 | -2.14 | . 118 | . 484 | . 0854 | . 665 | . 0875 |
| Canada 2006 | -1.35 | . 0902 | -1.21 | . 0903 | . 616 | . 0867 | . 826 | . 0874 |
| Canada 2008 | -2.20 | . 106 | -2.05 | . 107 | . 416 | . 0865 | . 639 | . 0863 |
| Canada 2011 | -1.14 | . 0816 | -. 918 | . 0819 | . 453 | . 0852 | . 749 | . 0870 |
| Canada 2015 | -. 320 | . 0746 | $-.104$ | . 0749 | . 209 | . 0860 | . 493 | . 0851 |
| Germany 2002 | . 495 | . 0686 | . 596 | . 0690 | . 145 | . 0841 | . 321 | . 0862 |
| Germany 2005 | . 0591 | . 0696 | . 186 | . 0700 | -. 265 | . 0851 | -. 0646 | . 0911 |
| Germany 2009 | -1.23 | . 0774 | $-1.01$ | . 0780 | -1.06 | . 0895 | -. 732 | . 119 |
| Germany 2021 | -2.85 | . 107 | -2.51 | . 108 | -2.64 | . 117 | -2.15 | . 0856 |
| Mexico 2006 | -. 331 | . 0700 | -. 160 | . 0706 | . 131 | . 0842 | . 399 | . 0883 |
| Mexico 2009 | -1.61 | . 0859 | -1.40 | . 0865 | -. 265 | . 0866 | . 0510 | . 0866 |
| Mexico 2012 | -. 707 | . 0705 | -. 511 | . 0711 | -. 457 | . 0851 | -. 154 | . 254 |
| AIC | 2776 |  | 2719 | 389 | 27432 |  | 2666 | 699 |

Note: outcomes are pairs $\left(F_{w i}, V_{i}-F_{w i}\right)$, rounded to integers: reference category is "no frauds;" incremental and extreme frauds counts are combined. Binomial regression model coefficient estimates with robust standard errors. $n=1183332$ polling stations. For the fixed effects Bangladesh 1991 is the reference category. ${ }^{a}$ Maximum absolute difference in $\pi_{2}$ between largest and smallest chain-specific posterior means. ${ }^{b}$ All-chains dip test $p$-value for $\pi_{2}$.

Table 22: Russia 2011 Duma (PR) Election Moscow eforensics Estimates

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :---: | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .918 | .88 | .944 |
|  | $\pi_{2}$ | Incremental Fraud | .0714 | .046 | .107 |
|  | $\pi_{3}$ | Extreme Fraud | .0106 | .00568 | .0152 |
| turnout | $\beta_{0}$ | (Intercept) | .526 | .490 | .563 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.284 | -.336 | -.242 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | .301 | .0919 | .447 |
|  | $\rho_{S 0}$ | (Intercept) | -.214 | -.497 | -.0324 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | .333 | .0178 | .581 |
|  | $\delta_{S 0}$ | (Intercept) | -.467 | -.730 | -.0323 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.978 .^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0525 ; M\left(\pi_{2}\right)=.0495 ; M\left(\pi_{3}\right)=.00444 .{ }^{d}$
units eforensics-fraudulent: (136 incremental, 38 extreme, 3199 not fraudulent)
manufactured votes

$$
\begin{aligned}
& F_{t}=29167.6[25807.1,31052.2]^{e} \\
& \quad F_{t}=24663.2[21290.9,26082.7]^{e}
\end{aligned}
$$

$$
\text { extreme manufactured } \quad F_{t}=4504.4[3507.5,5202.7]^{e}
$$

$$
\text { total eforensics-fraudulent votes } \quad F_{w}=51643.9[46446.9,54817.8]^{e}
$$

$$
\text { incremental total } \quad F_{w}=43355.8[38130.1,45847.2]^{e}
$$

$$
\text { extreme total } \quad F_{w}=8288.1[6659.7,9534.4]^{e}
$$

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). $n=3373$ units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=7181973 ; \sum_{i=1}^{n} V_{i}=4326522 ; \sum_{i=1}^{n} W_{i}=2052751$. ${ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Figure 18: eforensics-plots: Russia 2011 Duma, Moscow (a) original data

(b) original data: eforensics-fraudulent polling stations in red triangles; randomly observed polling stations in blue crosses; other polling stations in green circles


Note: For eforensics estimates see Table 22.

Figure 19: Scatterplot: Russia 2011 Duma, Moscow, with Randomly Observed and Neighoring Polling Stations
(a) original data: eforensics-fraudulent polling stations in red triangles and red asterisks; randomly observed polling stations in blue ' $x$ 's; neighboring polling stations in tan crosses and red asterisks; other polling stations in green circles


Note: For eforensics estimates see Table 22.

Table 23: Russia 2011 Duma (PR) Election Moscow eforensics Estimates, with Observer Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :---: | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .892 | .880 | .905 |
|  | $\pi_{2}$ | Incremental Fraud | .0970 | .0858 | .109 |
|  | $\pi_{3}$ | Extreme Fraud | .0110 | .00735 | .0147 |
| turnout | $\beta_{0}$ | (Intercept) | .571 | .527 | .6160 |
|  | $\beta_{1}$ | is observed | -.239 | -.297 | -.173 |
| vote choice | $\beta_{2}$ | is neighbor | -.153 | -.224 | -.106 |
|  | $\gamma_{0}$ | (Intercept) | -.261 | -.293 | -.227 |
| incremental frauds | $\gamma_{1}$ | is observed | -.390 | -.456 | -.335 |
|  | $\gamma_{2}$ | is neighbor | -.105 | -.163 | -.0630 |
| extreme frauds | $\rho_{M 0}$ | (Intercept) | .223 | .0598 | .336 |
|  | $\rho_{S 0}$ | (Intercept) | -.514 | -.671 | -.427 |
|  | $\delta_{M 0}$ | (Intercept) | .189 | -.0556 | .414 |
|  | $\delta_{S 0}$ | (Intercept) | -.694 | -1.02 | -.274 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=.834 ; D\left(\pi_{2}\right)=1 ; D\left(\pi_{3}\right)=.999 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.00853 ; M\left(\pi_{2}\right)=.00727 ; M\left(\pi_{3}\right)=.00126 .{ }^{d}$
units eforensics-fraudulent: (185 incremental, 38 extreme, 3150 not fraudulent)
manufactured votes $\quad F_{t}=38887.9[36224.0,44310.2]^{e}$
incremental manufactured $\quad F_{t}=34140.1[31776.3,38630.1]^{e}$
extreme manufactured $\quad F_{t}=4747.8[3797.6,5772.1]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=65338.9[62792.7,68484.6]^{e}$ incremental total $\quad F_{w}=56557.0[54551.9,58793.2]^{e}$ extreme total $\quad F_{w}=8782.0[7121.7,10377.1]^{e}$

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). $n=3373$ units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=7181973 ; \sum_{i=1}^{n} V_{i}=4326522 ; \sum_{i=1}^{n} W_{i}=2052751 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Figure 20: eforensics-plots: Argentina 2015 President Round 1
(a) original data

(b) departamentoresidualized data

Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables 24 and 25.

Table 24: Argentina 2015 President Election (Round 1) eforensics Estimates

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .990 | .980 | .999 |
|  | $\pi_{2}$ | Incremental Fraud | .00997 | .00101 | .0198 |
|  | $\pi_{3}$ | Extreme Fraud | .000181 | $1.65 \mathrm{e}-05$ | .000449 |
| turnout | $\beta_{0}$ | (Intercept) | 1.42 | 1.41 | 1.44 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.642 | -.661 | -.624 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.312 | -.631 | .0256 |
|  | $\rho_{S 0}$ | (Intercept) | -.247 | -.630 | .0510 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0660 | -.242 | .0268 |
|  | $\delta_{S 0}$ | (Intercept) | -.00646 | -.0713 | .0677 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=0 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0177 ; M\left(\pi_{2}\right)=.0175 ; M\left(\pi_{3}\right)=.000331 .{ }^{d}$
units eforensics-fraudulent: (31 incremental, 10 extreme, 92170 not fraudulent)
manufactured votes

$$
\begin{aligned}
& F_{t}=781.5[626.7,889.9]^{e} \\
& \quad F_{t}=428.5[259.4,602.5]^{e} \\
& F_{t}=353.0[215.5,492.6]^{e} \\
& F_{w}=2894.2[2333.2,3329.7355] \\
& \quad F_{w}=1568.3[935.4,2163.4]^{e} \\
& F_{w}=1325.9[810.6,1859.4]^{e}
\end{aligned}
$$

$$
\text { extreme manufactured } \quad F_{t}=353.0[215.5,492.6]^{e}
$$

$$
\text { total eforensics-fraudulent votes } \quad F_{w}=2894.2[2333.2,3329.7355]^{e}
$$

$$
\text { incremental total } \quad F_{w}=1568.3[935.4,2163.4]^{e}
$$

extreme total

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). $n=92204$ mesa units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=31164077 ; \sum_{i=1}^{n} V_{i}=24420841 ; \sum_{i=1}^{n} W_{i}=9002242 .{ }^{a} 95 \% \mathrm{HPD}$ lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Table 25: Argentina 2015 President Election (Round 1) eforensics Estimates, Departamento Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .995 | .994 | .997 |
|  | $\pi_{2}$ | Incremental Fraud | .00480 | .00336 | .00600 |
|  | $\pi_{3}$ | Extreme Fraud | $6.85 \mathrm{e}-05$ | $1.98 \mathrm{e}-05$ | .000124 |
| turnout | $\beta_{0}$ | (Intercept) | 1.39 | 1.366 | 1.42 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.636 | -.659 | -.606 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.131 | -.245 | -.0475 |
|  | $\rho_{S 0}$ | (Intercept) | -.798 | -.906 | -.692 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0594 | -.126 | .0444 |
|  | $\delta_{S 0}$ | (Intercept) | -.0168 | -.0500 | .0135 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=1 ; D\left(\pi_{2}\right)=.826 ; D\left(\pi_{3}\right)=.998 .^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0015 ; M\left(\pi_{2}\right)=.0015 ; M\left(\pi_{3}\right)=6.4 \mathrm{e}-06 .{ }^{d}$
units eforensics-fraudulent: (150 incremental, 5 extreme, 92056 not fraudulent)
manufactured votes

$$
\begin{gathered}
F_{t}=2631.0[1593.5,3380.7]^{e} \\
F_{t}=2384.8[1362.5,3121.4]^{e} \\
F_{t}=246.3[154.2,275.1]^{e} \\
F_{w}=6326.1[3935.1,8226.6]^{e} \\
F_{w}=5671.2[3343.5,7508.9]^{e} \\
F_{w}=654.9[419.8,734.6]^{e}
\end{gathered}
$$

$$
\text { total eforensics-fraudulent votes } \quad F_{w}=6326.1[3935.1,8226.6]^{e}
$$

incremental total
extreme total
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). Departamento fixed effects for turnout and vote choice are not shown. $n=92204$ mesa units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=31164077 ; \sum_{i=1}^{n} V_{i}=24420841 ; \sum_{i=1}^{n} W_{i}=9002242 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [ $99.5 \%$ credible interval].

Figure 21: eforensics-plots: Argentina 2015 President Round 2
(a) original data

(b) departamentoresidualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables 26 and 28.

Table 26: Argentina 2015 President Election (Round 2) eforensics Estimates (a) no fixed effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .9999 | .9998 | .99999 |
|  | $\pi_{2}$ | Incremental Fraud | $3.22 \mathrm{e}-05$ | $4.77 \mathrm{e}-09$ | $9.28 \mathrm{e}-05$ |
|  | $\pi_{3}$ | Extreme Fraud | $6.31 \mathrm{e}-05$ | $4.21 \mathrm{e}-06$ | .000165 |
| turnout | $\beta_{0}$ | (Intercept) | 1.37 | 1.35 | 1.38 |
| vote choice | $\gamma_{0}$ | (Intercept) | .0410 | .0171 | .0662 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.111 | -.251 | -.0103 |
|  | $\rho_{S 0}$ | (Intercept) | -.0565 | -.226 | .0486 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.00951 | -.0520 | .0355 |
|  | $\delta_{S 0}$ | (Intercept) | -.0103 | -.0581 | .0476 |

(b) turnout and vote choice Departamento fixed effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .977 | .902 | .9999 |
|  | $\pi_{2}$ | Incremental Fraud | .0230 | $3.05 \mathrm{e}-08$ | .0977 |
|  | $\pi_{3}$ | Extreme Fraud | .000193 | $1.78 \mathrm{e}-05$ | .000355 |
| turnout | $\beta_{0}$ | (Intercept) | 1.29 | 1.09 | 1.35 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.0104 | -.0573 | .0261 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.516 | -.694 | -.212 |
|  | $\rho_{S 0}$ | (Intercept) | -.845 | -.960 | -.718 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0828 | -.173 | -.0195 |
|  | $\delta_{S 0}$ | (Intercept) | -.0694 | -.155 | .0784 |

posterior multimodality diagnostics:
(a) no fixed effects:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=.385 ; D\left(\pi_{2}\right)=1 ; D\left(\pi_{3}\right)=.885 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.000121 ; M\left(\pi_{2}\right)=2.52 \mathrm{e}-05 ; M\left(\pi_{3}\right)=.0001 .^{d}$
(b) turnout and vote choice Departamento fixed effects:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=0$. ${ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0913 ; M\left(\pi_{2}\right)=.0914 ; M\left(\pi_{3}\right)=.000246{ }^{d}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). (b) Departamento fixed effects for turnout and vote choice are not shown. (a, b) $n=92632$ mesa units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=31569917 ; \sum_{i=1}^{n} V_{i}=24691223 ; \sum_{i=1}^{n} W_{i}=12711629 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means.

Table 27: Argentina 2015 President Election (Round 2) eforensics-fraudulent Vote Estimates
(a) no fixed effects:
units eforensics-fraudulent: (0 incremental, 2 extreme, 92630 not fraudulent)
manufactured votes $\quad F_{t}=75.5[45.7,110.0]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=217.4[133.6,313.8]^{a}$
(b) turnout and vote choice Departamento fixed effects:
units eforensics-fraudulent: (0 incremental, 19 extreme, 92613 not fraudulent)
manufactured votes $\quad F_{t}=726.6[445.3,982.9]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=1486.2[755.2,2058.2]^{a}$
Note: eforensics model fraudulent vote count estimates. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=31569917 ; \sum_{i=1}^{n} V_{i}=24691223 ; \sum_{i=1}^{n} W_{i}=12711629 .{ }^{a}$ posterior mean [99.5\% credible interval].

Table 28: Argentina 2015 President Election (Round 2) eforensics Estimates, Departamento Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .538 | .500 | .673 |
|  | $\pi_{2}$ | Incremental Fraud | .462 | .327 | .500 |
|  | $\pi_{3}$ | Extreme Fraud | .000170 | $1.40 \mathrm{e}-05$ | .000415 |
| turnout | $\beta_{0}$ | (Intercept) | .974 | .789 | 1.22 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.192 | -.232 | -.120 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.161 | -.846 | 1.21 |
|  | $\rho_{S 0}$ | (Intercept) | -2.06 | -2.69 | -1.33 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0867 | -.173 | .00198 |
|  | $\delta_{S 0}$ | (Intercept) | -.124 | -.240 | .0214 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.135 .^{c}$
posterior means difference $M\left(\pi_{1}\right)=.152 ; M\left(\pi_{2}\right)=.152 ; M\left(\pi_{3}\right)=.000311 .^{d}$
units eforensics-fraudulent: (48225 incremental, 7 extreme, 44407 not fraudulent)
manufactured votes

$$
\begin{gathered}
F_{t}=1088040.9[438939.5,1546455.4]^{e} \\
\quad F_{t}=1087677.9[438588.4,1545902.5]^{e} \\
\quad F_{t}=363.0[191.8,550.7]^{e} \\
F_{w}=1450675.7[985958.0,1829807.8]^{e} \\
\quad F_{w}=1449940.8[985191.1,1828730.3]^{e} \\
\quad F_{w}=734.9[316.5,1107.6]^{e}
\end{gathered}
$$

total eforensics-fraudulent votes
incremental total extreme total

Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). Departamento fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown. $n=92589$ mesa units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=31569917 ; \sum_{i=1}^{n} V_{i}=24691223$;
$\sum_{i=1}^{n} W_{i}=12711629 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains.
${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Figure 22: Argentina 2015 President Round 2: eforensics-frauds Magnitude Fixed Effect Parameters


Note: active fixed effects parameters (posterior means and $95 \%$ HPD intervals) for frauds magnitude ( $\rho_{M j}, \rho_{S j}, \delta_{M j}, \delta_{S j}$ ) parameters in the eforensics model reported in Table 28. Departamentos with extreme eforensics-frauds are: 22 San Luis small, 49 Tucumán small, 82 Catamarca small, 84 Córdoba small, 89 Córdoba Colón, 96 Entre Ríos small, 99 Formosa small. ("small" comprises all Departamentos each of which has fewer than 350 mesas in a Provincia).

Figure 23: eforensics-plots: Florida 2000 President
(a) original data

(b) county-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables ?? and ??.

Table 29: Florida 2000 President eforensics Estimates, County Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .857 | .843 | .871 |
|  | $\pi_{2}$ | Incremental Fraud | .143 | .129 | .157 |
|  | $\pi_{3}$ | Extreme Fraud | .000211 | $9.62 \mathrm{e}-08$ | .000639 |
| turnout | $\beta_{0}$ | (Intercept) | .672 | .634 | .705 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.203 | -.247 | -.158 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.402 | -.437 | -.355 |
|  | $\rho_{S 0}$ | (Intercept) | -.425 | -.485 | -.342 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | .00237 | -.124 | .145 |
|  | $\delta_{S 0}$ | (Intercept) | -.0557 | -.156 | .0380 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $D\left(\pi_{1}\right)=.999 ; D\left(\pi_{2}\right)=.991 ; D\left(\pi_{3}\right)=1 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0153 ; M\left(\pi_{2}\right)=.0155 ; M\left(\pi_{3}\right)=.000187 .{ }^{d}$
units eforensics-fraudulent: (389 incremental, 0 extreme, 5552 not fraudulent)
manufactured votes $\quad F_{t}=39310.5[36821.6,43453.4]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=85359.7[81093.6,90878.8]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). County fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown (see Figure 24). $n=5941$ precinct units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=8744117 ; \sum_{i=1}^{n} V_{i}=5961147$; $\sum_{i=1}^{n} W_{i}=2911796 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [ $99.5 \%$ credible interval].

Figure 24: Florida 2000 President: eforensics-frauds Magnitude Fixed Effect Parameters


Note: active fixed effects parameters (posterior means and $95 \%$ HPD intervals) for frauds magnitude ( $\rho_{M j}, \rho_{S j}$ ) parameters in the eforensics model reported in Table 29. Counties with eforensics-frauds are: 0 Alachua, 2 Baker, 3 Bay, 5 Brevard, 6 Broward, 7 Calhoun, 8 Charlotte, 10 Clay, 11 Collier, 15 Duval, 16 Escambia, 17 Flagler, 20 Gilchrist, 22 Gulf, 24 Hardee, 25 Hendry, 26 Hernando, 27 Highlands, 28 Hillsborough, 29 Holmes, 30 Indian River, 31 Jackson, 33 Lafayette, 34 Lake, 35 Lee, 36 Leon, 37 Levy, 40 Manatee, 41 Marion, 42 Martin, 43 Miami-Dade, 44 Monroe, 45 Nassau, 46 Okaloosa, 48 Orange, 49 Osceola, 50 Palm Beach, 51 Pasco, 52 Pinellas, 53 Polk, 54 Putnam, 55 Santa Rosa, 56 Sarasota, 57 Seminole, 58 St. Johns, 59 St. Lucie, 61 Suwannee, 64 Volusia, 66 Walton, 67 Washington.

Table 30: Wisconsin 2016 President Elections eforensics Estimates (a) Intercepts only:

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .823 | .645 | .938 |
|  | $\pi_{2}$ | Incremental Fraud | .177 | .0616 | .355 |
|  | $\pi_{3}$ | Extreme Fraud | .000304 | $3.86 \mathrm{e}-08$ | .000912 |
| turnout | $\beta_{0}$ | (Intercept) | 1.34 | 1.28 | 1.39 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.319 | -.528 | -.179 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.705 | -.991 | -.484 |
|  | $\rho_{S 0}$ | (Intercept) | .0411 | -.258 | .357 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.167 | -.314 | -.0436 |
|  | $\delta_{S 0}$ | (Intercept) | -.201 | -.469 | .0592 |

(b) Including county fixed effects: ${ }^{c}$

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .807 | .724 | .929 |
|  | $\pi_{2}$ | Incremental Fraud | .193 | .0710 | .276 |
|  | $\pi_{3}$ | Extreme Fraud | .000477 | $5.72 \mathrm{e}-08$ | .00145 |
| turnout | $\beta_{0}$ | (Intercept) | 1.38 | 1.345 | 1.43 |
| vote choice | $\gamma_{0}$ | (Intercept) | .0328 | -.0171 | .0992 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.452 | -1.20 | .355 |
|  | $\rho_{S 0}$ | (Intercept) | -1.02 | -1.28 | -.864 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.0125 | -.512 | .220 |
|  | $\delta_{S 0}$ | (Intercept) | -.157 | -.495 | .173 |

posterior multimodality diagnostics:
no fixed: all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=1$. ${ }^{d}$
posterior means difference $M\left(\pi_{1}\right)=.275 ; M\left(\pi_{2}\right)=.275 ; M\left(\pi_{3}\right)=.0000525 .{ }^{e}$
fixed eff.: all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=0 ; D\left(\pi_{2}\right)=0 ; D\left(\pi_{3}\right)=.972 .{ }^{d}$
posterior means difference $M\left(\pi_{1}\right)=.181 ; M\left(\pi_{2}\right)=.181 ; M\left(\pi_{3}\right)=.00068 .{ }^{e}$
(a) units eforensics-fraudulent: (240 incremental, 0 extreme, 3154 not fraudulent)
manufactured votes $\quad F_{t}=5618.8[2406.1,8120.8]^{a}$
total eforensics-fraudulent votes $\quad F_{w}=24721.9[11146.4,33979.2]^{a}$
(b) units eforensics-fraudulent: (199 incremental, 0 extreme, 3195 not fraudulent)
manufactured votes $\quad F_{t}=7788.9[2887.6,13025.8]^{e}$
total eforensics-fraudulent votes $\quad F_{w}=16438.6[11033.6,21840.1]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). $n=3394$ ward units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=3721467 ; \sum_{i=1}^{n} V_{i}=2971244 ; \sum_{i=1}^{n} W_{i}=1402592 .{ }^{a} 95 \% \mathrm{HPD}$ lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ County fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown (see Figure 25). ${ }^{d}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{e}$ difference between largest and smallest chain-specific posterior means.

Figure 25: Wisconsin 2016 President: eforensics-frauds Magnitude Fixed Effect Parameters


Note: active fixed effects parameters (posterior means and $95 \%$ HPD intervals) for frauds magnitude ( $\rho_{M j}, \rho_{S j}, \delta_{M j}, \delta_{S j}$ ) parameters in the eforensics model reported in Table 30. Counties with eforensics-frauds are: 0 Adams, 3 Barron, 4 Bayfield, 5 Brown, 7 Burnett, 8 Calumet, 9 Chippewa, 10 Clark, 11 Columbia, 13 Dane, 14 Dodge, 15 Door, 16 Douglas, 19 Florence, 20 Fond du Lac, 21 Forest, 22 Grant, 23 Green, 24 Green Lake, 26 Iron, 27 Jackson, 28 Jefferson, 30 Kenosha, 31 Kewaunee, 32 La Crosse, 33 Lafayette, 34 Langlade, 35 Lincoln, 36 Manitowoc, 37 Marathon, 41 Milwaukee, 42 Monroe, 43 Oconto, 44 Oneida, 45 Outagamie, 46 Ozaukee, 48 Pierce, 49 Polk, 50 Portage, 51 Price, 52 Racine, 54 Rock, 55 Rusk, 58 Shawano, 59 Sheboygan, 60 St Croix, 61 Taylor, 64 Vilas, 65 Walworth, 67 Washington, 68 Waukesha, 69 Waupaca, 70 Waushara, 71 Winnebago, 72 Wood.

Figure 26: eforensics-plots: Ohio 2004 President
(a) original data

(b) county-residualized data


Note: scatterplots, 2D empirical densities and marginal histograms for turnout and leader vote proportions. For eforensics estimates see Tables ?? and ??.

Table 31: Ohio 2004 President Election eforensics Estimates, County Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :--- | :--- |
| mixture probabilities | $\pi_{1}$ | No Fraud | .774 | .762 | .788 |
|  | $\pi_{2}$ | Incremental Fraud | .225 | .210 | .237 |
|  | $\pi_{3}$ | Extreme Fraud | .000702 | .000149 | .00137 |
| turnout | $\beta_{0}$ | (Intercept) | .797 | .776 | .823 |
| vote choice | $\gamma_{0}$ | (Intercept) | -.0427 | -.0861 | .0113 |
| incremental frauds | $\rho_{M 0}$ | (Intercept) | -.429 | -.462 | -.399 |
|  | $\rho_{S 0}$ | (Intercept) | -.403 | -.461 | -.358 |
| extreme frauds | $\delta_{M 0}$ | (Intercept) | -.322 | -.557 | -.150 |
|  | $\delta_{S 0}$ | (Intercept) | -.232 | -.383 | -.106 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=.981 ; D\left(\pi_{2}\right)=.998 ; D\left(\pi_{3}\right)=1 .{ }^{c}$
posterior means difference $M\left(\pi_{1}\right)=.0153 ; M\left(\pi_{2}\right)=.0160 ; M\left(\pi_{3}\right)=.000623 .^{d}$
units eforensics-fraudulent: (1924 incremental, 5 extreme, 9435 not fraudulent)
manufactured votes $\quad F_{t}=79378.2[73570.1,83534.4]^{e}$
incremental manufactured $\quad F_{t}=78920.2[73082.0,83153.5]^{e}$
extreme manufactured
$F_{t}=458.0[281.5,550.1]^{e}$
total eforensics-fraudulent votes
$F_{w}=177874.2[166201.6,186814.6]^{e}$
$F_{w}=176753.1[165035.9,185860.4]^{e}$
incremental total
$F_{w}=1121.1[710.5,1329.1]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). County fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown (see Figure 27). $n=11364$ precinct units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=7972292 ; \sum_{i=1}^{n} V_{i}=5411161$; $\sum_{i=1}^{n} W_{i}=2766860 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].

Figure 27: Ohio 2004 President: eforensics-frauds Magnitude Fixed Effect Parameters


Note: active fixed effects parameters (posterior means and $95 \%$ HPD intervals) for frauds magnitude ( $\rho_{M j}, \rho_{S j}, \delta_{M j}, \delta_{S j}$ ) parameters in the eforensics model reported in Table XX. Counties with extreme eforensics-frauds are: 32 Hamilton, 75 Shelby.

Table 32: Ohio 2004 President Election eforensics Estimates, African American Proportion Effects and County Fixed Effects

| Type | Parameter | Covariate | Mean | $\mathrm{lo}^{a}$ | $\mathrm{up}^{b}$ |
| :--- | :---: | ---: | :--- | :---: | :---: |
| mixture probabilities | $\pi_{1}$ | No Fraud | .830 | .817 | .847 |
|  | $\pi_{2}$ | Incremental Fraud | .169 | .153 | .182 |
|  | $\pi_{3}$ | Extreme Fraud | .000693 | .000162 | .00133 |
| turnout | $\beta_{0}$ | (Intercept) | .496 | .476 | .514 |
|  | $\beta_{1}$ | African American | -.0631 | -.0656 | -.0604 |
| vote choice | (Intercept) | -1.11 | -1.16 | -1.08 |  |
| incremental frauds | $\gamma_{0}$ | $\rho_{1}$ | African American | -.220 | -.225 |
|  | (Intercept) | -.155 | -.308 | -.0856 |  |
|  | $\rho_{M 1}$ | African American | .0572 | .0408 | .0720 |
|  | $\rho_{S 0}$ | (Intercept) | -.424 | -.523 | -.330 |
| extreme frauds | $\rho_{S 1}$ | African American | .0764 | .0707 | .0840 |
|  | $\delta_{M 0}$ | (Intercept) | -.298 | -.494 | -.100 |
|  | $\delta_{M 1}$ | African American | .101 | .0311 | .249 |
|  | $\delta_{S 0}$ | (Intercept) | -.307 | -.496 | -.124 |
|  | $\delta_{S 1}$ | African American | .0887 | .0402 | .140 |

posterior multimodality diagnostics:
all-chains dip test $p$-values $\quad D\left(\pi_{1}\right)=.038 ; D\left(\pi_{2}\right)=.020 ; D\left(\pi_{3}\right)=1 .{ }^{c}$
posterior means difference $\quad M\left(\pi_{1}\right)=.0180 ; M\left(\pi_{2}\right)=.0181 ; M\left(\pi_{3}\right)=.000552 .^{d}$
units eforensics-fraudulent: (1421 incremental, 6 extreme, 9804 not fraudulent)
manufactured votes incremental manufactured
$F_{t}=59753.0[50806.6,68715.8]^{e}$
$F_{t}=59395.5[50311.2,68163.8]^{e}$
extreme manufactured
$F_{t}=357.5[88.7,561.9]^{e}$
total eforensics-fraudulent votes
incremental total
$F_{w}=117786.3[97332.5,133960.6]^{e}$
$F_{w}=117038.9[96296.5,132793.0]^{e}$
extreme total
$F_{w}=747.4[184.4,1186.7]^{e}$
Note: selected eforensics model parameter estimates (posterior means and highest posterior density credible intervals). "African American" denotes the logit of the precinct proportion African American: (min, median, Q3, max) $=(-9.2,-4.4,-2.8,9.2)$. County fixed effects for turnout, vote choice and eforensics-frauds magnitudes are not shown. $n=11231$ precinct units. Electors, valid votes and votes for the leader: $\sum_{i=1}^{n} N_{i}=7900002 ; \sum_{i=1}^{n} V_{i}=5362107 ; \sum_{i=1}^{n} W_{i}=2738640 .{ }^{a} 95 \%$ HPD lower bound. ${ }^{b} 95 \%$ HPD upper bound. ${ }^{c}$ dip test for unimodality null hypothesis (Hartigan and Hartigan 1985) over all MCMC chains. ${ }^{d}$ difference between largest and smallest chain-specific posterior means. ${ }^{e}$ posterior mean [99.5\% credible interval].


[^0]:    1 "Elector" refers to a registered or otherwise eligible voter.

[^1]:    ${ }^{2}$ Klimek et al. (2012) rely on Borghesi and Bouchaud (2010), whose diffusion model represents local dependencies but not the kinds of global dependencies frauds induce.

[^2]:    ${ }^{3}$ The exponential hyperprior for $\sigma^{\kappa \xi}$ imposes some regularization.

[^3]:    ${ }^{4}$ At least one bureaux is created for every 300 electors (Ministère de l'Intérieur 2021).

[^4]:    ${ }^{5}$ Annulments models use a cases model with regressor $M$ to adjust for censoring (Maddala 1983, 277-278).
    ${ }^{6}$ Translation of paragraph 9 in Conseil Constitutionnel (2017).
    ${ }^{7}$ Column (d) in Table 2(A) illustrates the kind of multicollinearity that otherwise occurs, with coefficients having opposite signs.

[^5]:    ${ }^{8}$ To account for variation across MCMC draws, in Table 2(b-g) we use Normal approximation coefficient means and confidence intervals (Tanner 1986). For each MCMC draw the algorithm uses a robust covariance matrix, which among other things adjusts for overdisperion in the binomial regression models.

[^6]:    ${ }^{9}$ Implications are similar if instead of $D_{i}(6)$ includes $D_{i j}$ with precinct mean $D_{i}$. In the spirit of the Borghesi and Bouchaud (2010) diffusion model for local associations, correlations among terms $D_{i j}$ that occur only for individuals within each aggregation unit have essentially no implications for eforensics estimates given only aggregated (e.g. precinct-level) data.

[^7]:    ${ }^{10}$ In practice it is often advantageous to include geographically defined fixed effects in $x_{i}^{\nu}$ and $x_{i}^{\tau}$.

[^8]:    ${ }^{11}$ With eforensics we typically use four chains each of which draws on a distinct type of pseudorandom number generator.

[^9]:    ${ }^{12}$ Magnifying the images in a viewer can produce similar effects.

[^10]:    ${ }^{13}$ Including such fixed effects in the specification for the Afghanistan 2009 election changes the results in important ways not discussed here.

[^11]:    ${ }^{14}$ Elections for which eforensics-plots are shown somewhere in this paper have nonmissing "Figure" entries in the last column.
    ${ }^{15}$ These include Florida 2000 President, Ohio 2004 President, Ohio 2006 U.S. Senate, South Africa 2014 National, Uganda 2006 President.

[^12]:    ${ }^{16}$ For example, for the Ohio 2006 U.S. Senate election there are zero estimated eforensics-frauds.
    17 "Son casillas para que las personas en tránsito puedan votar si están lejos de su sección electoral" (Central Electoral 2019).
    ${ }^{18}$ A polling station is a UIK, uchastkovaya izbiratelnaya komissiya.

[^13]:    ${ }^{19}$ The rescaled YES and NO proportions sum to 1.0.

[^14]:    ${ }^{20}$ Use of robust standard errors corrects for simple overdispersion.

[^15]:    ${ }^{21}$ Some polls appear with zero electors but a positive number of votes cast. For these we assign $N_{i}:=V_{i}$ and mark the poll for the referent fixed effect.

[^16]:    ${ }^{22}$ Estimates $F_{t i}$ and $F_{w i}$ for the polls in Guelph find little that is particularly noteworthy.

[^17]:    ${ }^{23} \mathrm{~A}$ caveat is that for all fixed effects except any displayed in position zero, which corresponds to the intercept, I simply add the posterior mean of the intercept to the fixed effects' coefficient and to the limits of its $95 \%$ HPD interval, without adjusting for how these intervals should change to represent the full variation of the combined fixed effects. So pending implementation of such corrected credible intervals, the displays in Figure 16 should be viewed merely as informally illustrative.

[^18]:    ${ }^{24}$ The included polling stations are those from region Gorod Moskva in the polling station count data obtained from http://www.vybory.izbirkom.ru on December 11, 2011.

[^19]:    ${ }^{25}$ In each province I combine all departamentos with fewer than 350 electors into a set that I treat as a "small" departamento.

[^20]:    ${ }^{26}$ Recall the caveat in note 23 , which also applies here.

[^21]:    ${ }^{27}$ Technically the "African American" variable is the logit of the precinct proportion of electors who are African American, where the observed proportions are censored into the interval [.0001, .9999] before computing the logit.

