Chapter 2. Exercises

1. In the theory of relativity, space and time variables can be combined to form a 4-dimensional vector thus: \( x_1 = x, x_2 = y, x_3 = z, x_4 = ict \). The momentum and energy analogously combine to a 4-vector with \( p_1 = p_x, p_2 = p_y, p_3 = p_z, p_4 = iE/c \). By a suitable generalization of the quantization prescription for momentum components, deduce the time-dependent Schrödinger equation:

\[
\left\{-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right\} \Psi(r, t) = i\hbar \frac{\partial \Psi(r, t)}{\partial t}
\]

2. Estimate the number of photons emitted per second by a 100-watt light-bulb. Assume a wavelength of 550 nm (yellow light).

3. Electron diffraction makes use of 40 keV (40,000 eV) electrons. Calculate their de Broglie wavelength.

4. Show that the wavefunction \( \Psi(x, t) = e^{i(px-Et)/\hbar} \) is a solution of the one-dimensional time-dependent Schrödinger equation.

5. Show that \( \Psi(r, t) = e^{i(p \cdot r-Et)/\hbar} \) is a solution of the three-dimensional time-dependent Schrödinger equation.

6. A certain one-dimensional quantum system in \( 0 \leq x \leq \infty \) is described by the Hamiltonian:

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{q^2}{x} \quad (q = \text{constant})
\]

One of the eigenfunctions is known to be

\[
\psi(x) = Ax e^{-\alpha x}, \quad \alpha \equiv \frac{mq^2}{\hbar^2}, \quad A = \text{constant}
\]

(i) Write down the Schrödinger equation and carry out the indicated differentiation.
(ii) Find the corresponding energy eigenvalue (in terms of $\hbar$, $m$ and $q$).

(iii) Find the value of $A$ which normalizes the wavefunction according to

$$\int_{0}^{\infty} |\psi(x)|^2 dx = 1$$

You may require the definite integrals

$$\int_{0}^{\infty} x^n e^{-ax} dx = n!/a^{n+1}$$
Answers to Exercises

Don’t even think of looking here before you attempt to solve the problems yourself!

1. The components of the momentum operator can be expressed in the form

\[ \hat{p}_k = -i\hbar \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3 \]

Now extend this relation for \( k = 4 \) using \( p_4 = iE/c \) and \( x_4 = ict \). The result is

\[ \hat{H} = +i\hbar \frac{\partial}{\partial t} \]

where the energy operator is the Hamiltonian \( \hat{H} \). Applying the quantization prescription to the classical energy-momentum relation

\[ E = \frac{p^2}{2m} + V(x, y, z) \quad p^2 = p_1^2 + p_2^2 + p_3^2 \]

then leads to the 3-dimensional time-dependent Schrödinger equation (29).

2. 100 watts = 100 J/sec. The energy of a 550 nm photon is given by

\[ E = h\nu = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{550 \times 10^{-9}} = 3.61 \times 10^{-19} \text{ J} \]

Thus 100/E = 2.77 \times 10^{20} photons/sec.

3. Since 1 eV=1.602 \times 10^{-19} \text{ J}, each electron has a kinetic energy of \((40 \times 10^3)(1.602 \times 10^{-19})\) J. This is equal to

\[ E = \frac{1}{2}mv^2 = \frac{p^2}{2m} \]

The de Broglie relation \( \lambda = h/p \), therefore gives

\[ \lambda = \frac{h}{\sqrt{2mE}} = \frac{6.626 \times 10^{-34}}{\sqrt{2(9.109 \times 10^{-31})(40 \times 10^3)(1.602 \times 10^{-19})}} \]
= 6.13 × 10^{-12} \text{ m}. This gives sufficient resolution to study the geometric structure of molecules. [Since 40 keV electrons travel at a significant fraction of the speed of light, the relativistic energy-momentum relation must be used. The corrected de Broglie wavelength is actually 6.016 × 10^{-12} \text{ m}].

4. Evaluate the partial derivatives

\[ \frac{\partial}{\partial x} \Psi(x, t) = \frac{i}{\hbar} p_x e^{i(px-Et)/\hbar} \]
\[ \frac{\partial^2}{\partial x^2} \Psi(x, t) = -\frac{p_x^2}{\hbar^2} e^{i(px-Et)/\hbar} \]

and \[ \frac{\partial}{\partial t} \Psi(x, t) = -\frac{iE}{\hbar} e^{i(px-Et)/\hbar} \]

Eq (26) then follows from the relation \( E = p^2/2m \).

5. Note that \( \mathbf{p} \cdot \mathbf{r} = p_x x + p_y y + p_z z \). Then

\[ \frac{\partial}{\partial x} \Psi(r, t) = \frac{i}{\hbar} p_x e^{i(p \cdot r - Et)/\hbar} \]

and Eq (29), with \( V(r)=0 \), follows from \( E = (p_x^2 + p_y^2 + p_z^2)/2m \).

6. Evaluate the derivatives (suppressing \( A \) for now):

\[ \psi'(x) = e^{-\alpha x} - \alpha x e^{-\alpha x} \quad \text{and} \quad \psi''(x) = -2\alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x} \]

Then the Schrödinger equation \( \hat{H} \psi(x) = E \psi(x) \) becomes

\[ -\frac{\hbar^2}{2m} (-2\alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x}) - \frac{q^2}{\hbar^2} x e^{-\alpha x} = E x e^{-\alpha x} \]

Now, cancel out the \( e^{-\alpha x} \) and find two independent relations for the terms independent of \( x \) and linear in \( x \). The results give \( \alpha = mq^2/\hbar^2 \), which agrees with the definition and

\[ E = -\frac{\hbar^2 \alpha^2}{2m} = -\frac{mq^4}{2\hbar^2} \]

To normalize the function

\[ \int_0^\infty |\psi(x)|^2 \, dx = 1 = A^2 \int_0^\infty x^2 e^{-2\alpha x} \, dx = A^2 \times 2!/ (2\alpha)^3 \]

giving \( A = 2\alpha^{3/2} \).