RG Flows, Entanglement & Holography Workshop

Michigan Center for Theoretical Physics
September 17 – 21, 2012
Intersections in Theoretical Physics:

- Renormalization Group Flows
- Statistical Mechanics
- Many Body Theory
- Entanglement
- Quantum Information
- Holography
- Particle Physics
- Quantum Field Theory
- String Theory
- Condensed Matter Theory
- Quantum Gravity
- Quantum Field Theory
Renormalization Group

Wikipedia:
“mathematical apparatus that allows systematic investigation of the changes of a physical system as viewed at different distance scales”

Universality: physics at long distances is largely insensitive to details of physics at short distances

Why Physics Works!

NOT Our World:
12 seconds! The heck with loop quantum gravity. We’ll have to try a string theory approach.
Renormalization Group Flows

Wikipedia: “mathematical apparatus that allows systematic investigation of the changes of a physical system as viewed at different distance scales”

Effects of short-distance physics is absorbed in values of parameters of an effective field theory

**RG Flows** describe how parameters of a quantum field theory change as more and more degrees of freedom at different scales are methodically “integrated out”

fixed points: flow is stationary

“critical phenomena”

**Conformal Field Theory (CFT):** QFT with few dimensionless couplings; scale invariant theory
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Holography

- idea in quantum gravity has its origins in trying to understand paradoxes related to the realization that black holes emit (almost) thermal radiation

- Bekenstein-Hawking entropy:
  \[ S_{BH} = \frac{k_B c^3}{\hbar} \frac{A_{\text{hor}}}{4G_N} \]

  speculation that dynamics and evolution of black hole could be described in terms of a theory living on the horizon

  (‘t Hooft; Susskind)
AdS/CFT Correspondence:

**Bulk:**
- quantum gravity
- negative cosmological constant
- \(d+1\) spacetime dimensions

**Boundary:**
- quantum field theory
- no scale (at quantum level)
- \(d\) spacetime dimensions
- no gravity!

Holography

anti-de Sitter space

conformal field theory

radius \(\leftrightarrow\) energy

(Maldacena ‘97)
Holographic RG flows:

\[ I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] \]

- Imagine potential has stationary points giving negative cosmological constant
  \[ V(\phi_{i,cr}) = 2 \Lambda_i < 0 \]

- \( \phi = \phi_{i,cr} \) the gravity solution is AdS space – dual to a particular CFT

- Holographic RG flows are solutions starting at one stationary point at large radius and ending at another at small radius – connects two CFTs between high and low energies.
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Quantum Entanglement

Einstein-Podolsky-Rosen Paradox:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]

Quantum Information: entanglement becomes a resource for (ultra)fast computations and (ultra)secure communications

Condensed Matter: entanglement is key to “unusual” phenomena, e.g., quantum Hall fluids, unconventional superconductors, . . .
Entanglement Entropy

- general tool to measure of correlations between subsystems

**procedure:**
- divide system into two subsystems, eg, A and B
  - integrate out degrees of freedom in subsystem B
  - remaining dof in A are described by a density matrix $\rho_A$
  - calculate **von Neumann entropy:** $S_{EE} = -Tr[\rho_A \log \rho_A]$

- applied to QFT: typically introduce a (smooth) boundary or entangling surface $\Sigma$ which divides the Cauchy surface (ie, time slice) into two separate regions

$$(t = constant)$$
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$

  calculate von Neumann entropy: $S_{EE} = -Tr[\rho_A \log \rho_A]$

- result is UV divergent!

- must regulate calculation:

  \[
  S_{EE} = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots
  \]

  $\delta$ = short-distance cut-off
  $d$ = spacetime dimension

- careful analysis reveals geometric structure, eg, $S_{EE} = \tilde{c}_0 \frac{A_{\Sigma}}{\delta^{d-2}} + \cdots$

(Sorkin `85; Bombelli, Koul, Lee & Sorkin `86)
Entanglement Entropy

- remaining dof are described by a density matrix $\rho_A$
- calculate von Neumann entropy: $S_{EE} = -Tr[\rho_A \log \rho_A]$

\[(t = \text{constant})\]

- result is UV divergent!
- must regulate calculation:
  \[S_{EE} = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots\]
  \[\delta = \text{short-distance cut-off}\]
  \[d = \text{spacetime dimension}\]
- find universal information characterizing underlying QFT in subleading terms, eg,
  \[S_{EE} = \cdots + c_d \log \left(\frac{R}{\delta}\right) + \cdots\]
Holographic Entanglement Entropy:

(Ryu & Takayanagi)

looks like BH entropy!

AdS boundary

AdS bulk spacetime

$r = \infty$

conformal field theory

gravitational potential/redshift

$S(A) = \partial V = \Sigma \frac{A_V}{4G_N}$

looks like BH entropy!
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John Preskill (quant-ph/9904022)

Holographic RG Flows

Holographic Entanglement Entropy
Zamolodchikov c-theorem (1986):

• renormalization-group flows can seen as one-parameter motion:

\[ \frac{d}{dt} \equiv -\beta^i(g) \frac{\partial}{\partial g^i} \quad (t = \text{scale}; \quad \beta^i = \partial_t g^i) \]

in the space of (renormalized) coupling constants \( \{g^i, \ i = 1, 2, 3, \cdots\} \) with beta-functions as “velocities”

• for unitary, Lorentz-invariant, renormalizable QFT’s in **two dimensions**, there exists a positive-definite real function of the coupling constants: \( C(g) \)

  1. monotonically decreasing along RG flows: \( \frac{d}{dt} C(g) \leq 0 \)

  2. “stationary” at fixed points: \( g^i = (g^*)^i \)

      \[ \beta^i(g^*) = 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial g^i} C(g) = 0 \]

  3. at fixed points, it equals central charge of corresponding CFT

      \[ C'(g^*) = c \]
with Zamolodchikov's framework:

BECOMES
with Zamolodchikov's framework:

\[ C(g) \]

\[ C_{\text{UV}} \]

\[ C_{\text{IR}} \]

Consequence for any RG flow in d=2: \( C_{\text{UV}} > C_{\text{IR}} \)
RG flows Meet Entanglement:

- c-theorem for d=2 RG flows can be established using unitarity, Lorentz invariance and **strong subadditivity inequality**:

  \[ S(X \cup Y \cup Z) - S(X \cup Y) - S(Y \cup Z) + S(Y) \leq 0 \]

- define: \( C(\ell) = 3 \ell \partial_{\ell} S(\ell) \)

  \[ \partial_{\ell} C(\ell) \leq 0 \]

- for d=2 CFT: \( S = \frac{c}{3} \log(\ell/\delta) + a_0 \) (Calabrese & Cardy)

  \[ C_{\text{CFT}}(\ell) = c \]

- hence it follows that: \( C_{\text{UV}} > C_{\text{IR}} \)
C-theorems in higher dimensions??

\begin{align*}
d=2: & \quad \langle T_\mu^\mu \rangle = -\frac{c}{12} R \\
d=4: & \quad \langle T_\mu^\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2} \nabla^2 R
\end{align*}

where \( I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \) and \( E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \)

• in 4 dimensions, have three central charges: \( c, a, a' \)

• do any of these obey a similar “c-theorem” under RG flows? \( \text{ie, } [??]_{\text{UV}} > [??]_{\text{IR}} \)

\textit{a-theorem:} proposed by Cardy (1988)

• numerous nontrivial examples, eg, perturbative fixed points (Osborn ‘89), SUSY gauge theories (Anselmi et al ‘98; Intriligator & Wecht ‘03)

• holographic field theories with Einstein gravity dual (Freedman et al ‘99; Giradello et al ‘98)

• \textbf{counterexample found!} (Shapere & Tachikawa ‘08)
C-theorems in higher dimensions??

in fact, past two years have seen a resurgence and remarkable progress:

• generalized holographic models with higher curvatures (RM & Sinha ‘10)
  → identify C-function with calculation of entanglement entropy for any \( d \)
  → single out A-type trace anomaly for even dimensions (ie, a for \( d=4 \))
  → conjecture new c-theorems for odd dimensions

• counterexample to \( d=4 \) a-theorem removed (Gaiotto, Seiberg & Tachikawa ‘10)

• entanglement entropy approach equivalent to F-theorem (Casini, Huerta & RM ‘11)

• F-theorem: \( d=3 \) (or any odd \( d \)) c-theorem; “free energy” → C-function;
  evidence from SUSY and nonSUSY QFT’s (Jafferis, Klebanov, Pufu & Safdi ‘11)

• \( d=4 \) a-theorem proved! dilaton effective actions (Komargodski & Schwimmer ‘11)

• \( d=3 \) F-theorem proved! Lorentz invariance & subadditivity (Casini & Huerta ‘12)
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**Renormalization Group Flows**

- Statistical Mechanics
- "A-Z"-theorems
- Many Body Theory
- Quantum Information

**Entanglement**

- Condensed Matter Theory

**Holography**

- Holographic RG Flows

**Holographic Entanglement Entropy**
Questions:

• how much of Zamalodchikov’s structure for 2d RG flows lifts higher dimensions?

  ➔ gradient flow? C-function stationary at fixed points?
  ➔ beyond fixed points, eg, limit cycles?

• “I”-theorems for RG flows in higher dimensions, d>4? (I = your favorite letter goes here)

• what can entanglement entropy/quantum information really say about renormalization group?

• further lessons for RG flows and entanglement from holography?

• what can RG flows and entanglement teach us about AdS/CFT correspondence?

  ➔ translation of “null energy condition” to boundary theory?

• what is the role of entanglement in quantum structure of spacetime/quantum gravity?

  ➔ Hawking radiation, information paradox, “firewalls,” . . . .

Lots to explore!