Here’s how to take this preparatory exam: Without reference to books, calculators, do this preparatory exam in two hours on your own. Then, get together with classmates to grade your preparatory exam, and work out the correct answers. At least one question from this preparatory exam will appear on Exam 2. You do not need to turn in your reworked exam. It will not be graded.

**Typical exam instructions:**

This is a closed book/closed notes test, and a calculator is not permitted (nor required). The duration of this Exam is 2.0 hours.

There are 4 problems, with different levels of difficulty. However, they are all weighted the same. Please plan your time accordingly.

Feel free to attach sheets as necessary, but please label which problem they refer to.

Please sign this page to indicate that you understand the above instructions and that you have, and will, abide by the Honor Code and Policy for this class. You do not need to provide numerical answers. You are expected to quote your results in terms of equations and symbols.

The grades on this exam are assigned as follows (1/3) for correct interpretation and start-up, and (1/3) for correct manipulation of the machinery to solve the problem. The remainder of the grade is given to correct final mathematical development, and discussion of the solution.

1. **Basic questions on the one-dimensional Schrödinger equation**

   Answer the following in one or two sentences. Equations are not required.

   (a) For a single particle with mass $m$ and energy $E$ in a one-dimensional potential $V(x)$, the conservation of energy in Newtonian (classical) mechanics is stated as:

   \[
   \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + V(x) = E.
   \]

   What is the equivalent statement in Quantum mechanics? What is the difference between them, and why does that difference exist?

   (b) Newton’s laws can be solved to give the exact future behavior of a particle. In what sense does the Schrödinger equation do this? In what sense does it not?

   (c) Why is it important for a wave function to be normalized? Is an unnormalized wave function a solution to the Schrödinger equation?

   (d) Discuss the physical meaning of \( \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1 \).

   Discuss the physical meaning of \( \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx \).

   (e) In the double-slit experiment, electrons that pass through the slits exhibit an interference pattern, even when they are known to pass through one at a time. How can you interpret this?

   (f) The energies of the excited states in the binding potentials we have discussed so far in the course, are exact. What does this suggest about the lifetimes of these excited states? Left on its own, will a particle ever make a transition from an excited state to a lower state?

   (g) Explain how a particle in a one-dimensional infinite potential well, can be considered to be a standing de Broglie wave.
2. **Superposition**

This problem concerns the concept of superposition, adding more than one wavefunction together. Consider the two composite wavefunctions, $u_a = N_a[a_1 u_1(x) \exp(-i(E_1/\hbar)t) + a_2 u_2(x) \exp(-i(E_2/\hbar)t)]$ and $u_b = N_b[b_1 u_1(x) \exp(-i(E_1/\hbar)t) + b_2 u_2(x) \exp(-i(E_2/\hbar)t)]$ where the $a_i$’s and $b_i$’s are fixed and known, and $u_1$ and $u_2$ are orthonormal (but otherwise arbitrary) eigenfunctions for some potential.

(a) Normalize $u_a$ and $u_b$.

(b) Find $\langle u_a | u_b \rangle$ and $\langle u_b | u_a \rangle$ and thereby conclude that $\langle u_b | u_a \rangle = \langle u_a | u_b \rangle^*$. 

(c) Find $\langle u_a | x | u_a \rangle$, $\langle u_a | p_x | u_a \rangle$, and $\langle u_a | E | u_a \rangle$, for the special case where $a_1 = a_2$.

3. **The box potential**

This problem concerns the basic properties of the solutions to the box potential.

(a) Show that $\langle m | n \rangle = \delta_{m,n}$. This means that the wavefunctions are orthonormal. Hint: It’s a lot easier to show if you use the exponential form, $\sin z = (e^{iz} - e^{-iz})/(2i)$.

(b) Find the mean and mean-square position of the $n$’th wavefunction: $\langle n | x | n \rangle$ and $\langle n | x^2 | n \rangle$.

(c) Find the mean and mean-square momentum of the $n$’th wavefunction: $\langle n | p_x | n \rangle$ and $\langle n | p_x^2 | n \rangle$. 

Hint: $\langle n | p_x | n \rangle = \langle n | -i\hbar \frac{\partial}{\partial x} | n \rangle$, $\langle n | p_x^2 | n \rangle = \langle n | -\hbar^2 \frac{\partial^2}{\partial x^2} | n \rangle$

(d) Find the variances in position and momentum of the $n$’th wavefunction: $(\Delta x)^2 = \langle n | x^2 | n \rangle - \langle n | x | n \rangle^2$ and $(\Delta p_x)^2 = \langle n | p_x^2 | n \rangle - \langle n | p_x | n \rangle^2$.

(e) Find $(\Delta p_x) / (\Delta x)$ and compare with the Heisenberg uncertainty relationships. What is the smallest $(\Delta p_x) / (\Delta x)$ possible in the box potential?

4. **The harmonic oscillator**

In class, we discussed and examined, the solutions to the Schrödinger equation for the 1D harmonic oscillator potential with spring constant $k$,

$$V(x) = \frac{kx^2}{2}.$$ 

The ground state wavefunction is given by:

$$u_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}} \exp[-\frac{1}{2}(\alpha x)^2]; \quad \alpha = \left(\frac{mk}{\hbar^2}\right)^{1/4}$$

For this ground state wave function, that has energy $E_0$:

(a) Using the Schrödinger equation,

$$-\frac{\hbar^2}{2m} u_0''(x) + V(x) u_0(x) = E_0 u_0(x),$$

determine the ground state energy $E_0$ in terms of $k$, $m$, and $\hbar$.

(b) Show that $\langle V \rangle = \langle \frac{1}{2} k x^2 \rangle = E_0/2$. 

(i) What does “parity” mean.

(h) In the solution to the infinite one-dimensional binding square well, we found that the eigenenergies took the form:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

where $L$ is the width of the well, $m$ is the mass of the particle, and $n = 1, 2, 3, \ldots$. Does this imply that we know the momentum of the particle exactly? Explain.
(c) Show that $\langle K \rangle = \left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) = E_0/2$, either explicitly or by using the result of part a) and the Conservation of Energy Law.

(d) Relate the above two results with the classical result shown in class. (You may do this part by memory, or re-derive the classical result.)

(e) Using the above results, show that:

$$\Delta p \Delta x = \sqrt{(\langle p^2 \rangle - \langle p \rangle^2)(\langle x^2 \rangle - \langle x \rangle^2)} = \hbar/2$$

(f) Discuss the significance of the above result with respect to the Heisenberg Uncertainty Principle.

You might find the following integrals useful.

$$\int_{-\infty}^{\infty} dz \ e^{-z^2} = \sqrt{\pi} \ ; \ \int_{-\infty}^{\infty} dz \ ze^{-z^2} = 0 \ ; \ \int_{-\infty}^{\infty} dz \ z^2 e^{-z^2} = \sqrt{\pi}/2$$