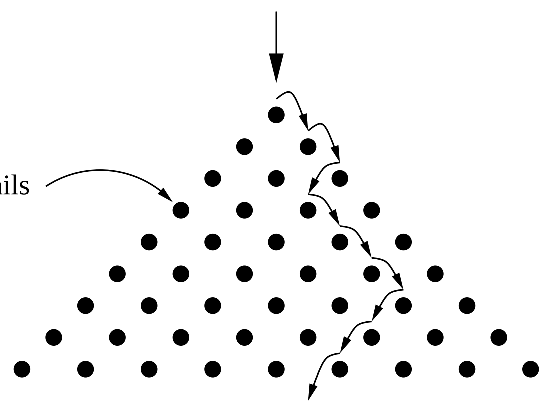
Physics 406: Homework 4

1. **Adiabatic compression of an ideal gas:**

   (a) Show that the adiabatic compressibility of the ideal gas satisfies $p\gamma\kappa_S = 1$ where γ is the ratio of the specific heats.

   (b) Later in the course we will demonstrate that $\gamma = \frac{5}{3}$ for an ideal gas. You can't show this using just thermodynamics—you need the Sackur–Tetrode equation, a statistical mechanical result that we'll get to in a couple of weeks. The important point however is to note that γ is a constant for the ideal gas. Hence show that the adiabatic compression of an ideal gas follows a curve $pV^\gamma = \text{constant}$.

2. **Combinatorics:** A famous carnival toy is a machine that drops a large number of marbles at the top of a pyramid of nails tacked to a board like this:



   The marbles fall down and at each step have a 50% chance of going right or left.

   (a) If there are N rows of nails total, write down an expression for the number of paths $g(N,l,r)$ that a marble can take that make a total of l steps to the left and r steps to the right. Eliminate l and r in favor of the distance $x = r - l$ traveled to the right to get the same number in terms of N and x only. (Note that the distance x is measured horizontally from where the marbles start, i.e., from a line down the middle of the picture above.)

   (b) If $N = 10$, how many ways are there of traveling distance $x = 10$ to the right? How many ways are there of traveling distance zero?

   (c) If $N = 10$, about how many marbles will have to be dropped before even a single one of them goes all the way to the right-most slot? And how many if $N = 20$?

   (d) When many marbles are dropped, what is the expected mean distance $\langle x \rangle$ traveled, averaged over all of them? And what is the standard deviation of the distance?

   (e) So if you had to say where a single marble dropped would land, between about which values of x would you feel reasonably confident saying it would end up, if $N = 100$? (If you want to be really precise, you could say which values would you have 90% confidence it would land between, but any sensible answer will do for this question. Saying that x lies between -100 and 100 is not a good answer!)

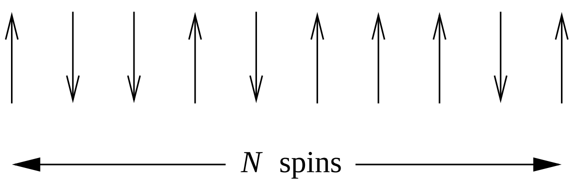
3. **Temperature:** We have seen that for a system of fixed volume and number of particles, the temperature τ and entropy σ are related by

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U},$$

where U is the internal energy. Thus we can calculate a rough value for the temperature of a system by comparing a small change ΔU in the internal energy with the corresponding change $\Delta\sigma$ in the entropy thus:

$$\tau = \frac{\Delta U}{\Delta\sigma}. \tag{1}$$

(a) For the system of magnetic "spins" (i.e., dipoles) that we discussed in class:



write down the internal energy U of the system in a magnetic field B (in the direction of the upward-pointing spins) in terms of the magnetic moment m of one of the dipoles, and the spin excess s defined by

$$2s = N_\uparrow - N_\downarrow,$$

where $N_\uparrow$ and $N_\downarrow$ are the numbers of spins pointing up and down respectively. Also write down the multiplicity $g(N, N_\uparrow)$ in terms of s. What is the entropy in terms of s?

(b) What is the change in energy when the spin excess goes from s to $s+1$? Show that the corresponding change in entropy is

$$\Delta\sigma = \ln \frac{\frac{1}{2}N - s}{\frac{1}{2}N + s + 1}.$$

(c) Calculate a rough expression for the temperature τ of the system from Eq. (1) above. What is the value of τ if $N = 20$, $s = 2$, and $m = 1$, $B = 1$?