Physics 406: Homework 7

1. **Expansion of a photon gas:** As we have seen, the partition function for a single electromagnetic mode of angular frequency ω in a box is

$$Z(\omega) = \frac{1}{1 - \exp(-\hbar\omega/\tau)}. \tag{1}$$

   (a) What is the free energy $F(\omega)$ of a single mode?

   (b) We have also shown that between frequencies ω and $\omega + \mathrm{d}\omega$ there are $V\omega^2\,\mathrm{d}\omega/\pi^2 c^3$ modes, where c is the speed of light. Given that the modes don't interact at all, what is the free energy F of the entire photon gas for all frequencies? You can leave the result in the form of an integral.

   (c) Change variables to $x = \hbar\omega/\tau$ to get an expression for F that involves ω, τ, $\hbar$, c, and the dimensionless integral

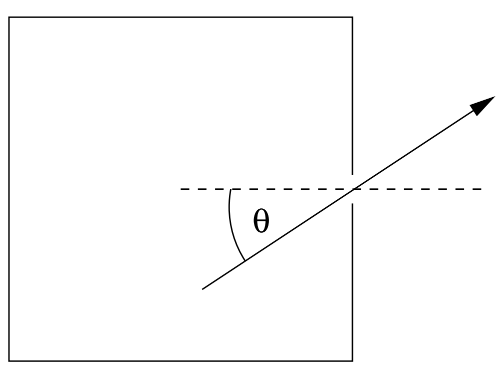
$$\int_0^\infty x^2 \ln(1 - \mathrm{e}^{-x})\,\mathrm{d}x = -\frac{\pi^4}{45}. \tag{2}$$

   Hence find a formula for the entropy of the gas.

   (d) If the gas expands adiabatically (i.e., at constant entropy) how are the temperature and volume related during the expansion?

   (e) The universe is full of radiation—the so-called microwave background discovered by Penzias and Wilson in 1965. Currently that radiation has a temperature of about 2.7K, but the universe is expanding. Assuming it expands adiabatically, how much smaller was the volume of the universe when the temperature of the background radiation was 27K?

2. **The Stefan-Boltzmann constant:** We have argued that a small hole in the side of a box at temperature T radiates as a black body at that temperature. Suppose the hole has area A and consider the radiation that escapes from the box through the hole:



We have shown that the internal energy of the radiation in the box is $U = (\pi^2 k^4 T^4 V)/(15\hbar^3 c^3)$, so the energy density is

$$\frac{U}{V} = \frac{\pi^2 k^4 T^4}{15\hbar^3 c^3},$$

where c is the speed of light and k is the Boltzmann constant. Consider the radiation incident on the hole from a direction that makes an angle θ with the normal to the hole, and arrives within a solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ of that direction.

(a) What is the energy in a small volume element $r^2\,\mathrm{d}r\,\mathrm{d}\Omega$, where r is measured from the hole? And what fraction of this actually hits the hole, rather than going in some other direction? Thus integrate over r from 0 to c to find the amount of energy transported through the hole per unit time from the given solid angle $\mathrm{d}\Omega$.

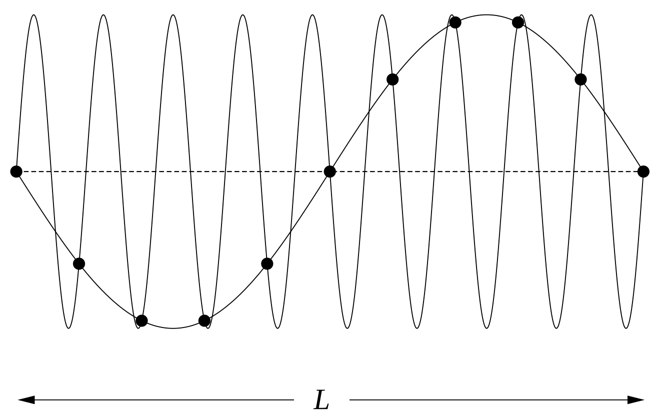
(b) Integrate over θ and ϕ to get the total radiation leaving the hole per unit time. (Hint: be careful about the limits of the integration.)

(c) The amount of radiation J per unit area given off by a black body at temperature T is given by the Stefan-Boltzmann law:

$$J = \sigma_B T^4,$$

where σ_B is the Stefan-Boltzmann constant. Show that the Stefan-Boltzmann constant has the value $5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.

3. **Aliasing:** Here's a picture representing the aliasing phenomenon that's much better than the one I drew in class:



It depicts a one-dimensional version of the aliasing problem. The dots are atoms. The solid lines represent two possible waves, both of which correspond to the same positions of the atoms.

Suppose the length of the system in atom spacings is L, as shown. In the picture $L = 10$. Let the atoms be numbered $k = 0 \ldots L$. Then the displacement y_k of the kth atom is

$$y_k = \sin(2\pi k/\lambda_h),$$

where λ_h is the wavelength of the high-frequency wave, which is related to the frequency by $\lambda_h \omega_h = 2\pi v$, with v being the speed of sound.

(a) Show that if $\lambda_h = 2L/n$, where n is any integer, then the wave is zero at both ends of the system. How many such modes are there between frequencies ω and $\omega + \mathrm{d}\omega$?

(b) Show that another similar wave with the longer wavelength $\lambda_l = 2L/(2L - n)$ gives the same displacements of the atoms (except for a minus sign).

(c) Hence what is the Debye frequency $\omega_{\max}$ for this one-dimensional solid (i.e., the maximum frequency of a unique wave in the system)?