

## Complex Systems 535/Physics 508: Homework 5

1. Consider a configuration model in which every vertex has the same degree  $k$ .
  - (i) What is the degree distribution  $p_k$ ? What are the generating functions  $g_0$  and  $g_1$  for the degree distribution and the excess degree distribution?
  - (ii) Show that the probability of a node belonging to the giant component is 1 for all  $k \geq 3$ , i.e., that the giant component is the size of the entire network, at least in the limit of large network size.
  - (iii) What happens when  $k = 1$ ?
  - (iv) **Extra credit:** What happens when  $k = 2$ ?
2.
  - (i) Let us model the Internet as a configuration model with a perfect power-law degree distribution  $p_k \sim k^{-\alpha}$ , with  $\alpha \simeq 2.5$  and  $k \geq 1$ . Write down the fundamental generating functions  $g_0$  and  $g_1$ .
  - (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they belong to the largest component).
3. Consider a model of a growing directed network similar to Price's model described in Section 14.1, but without preferential attachment. That is, vertices are added one by one to the growing network and each has  $c$  outgoing edges, but those edges now attach to existing vertices uniformly at random, without regard for degrees or any other vertex property.
  - (i) Derive master equations, the equivalent of Eqs. (14.7) and (14.8), that govern the distribution of in-degrees  $q$  in the limit of large network size.
  - (ii) Hence show that in the limit of large size the in-degrees have an exponential distribution  $p_q = Ce^{-\lambda q}$  with  $\lambda = \ln(1 + 1/c)$ .
4. Consider a model network similar to the model of Barabási and Albert described in Section 14.2, in which undirected edges are added between vertices according to a preferential attachment rule, but suppose that the network does not grow—it starts off with a given number  $n$  of vertices and neither gains nor loses any vertices thereafter. In this model, starting with an initial network of  $n$  vertices and some specified arrangement of edges, we add at each step one undirected edge between two vertices, both of which are chosen at random in direct proportion to degree  $k$ . Let  $p_k(m)$  be the fraction of vertices with degree  $k$  when the network has  $m$  edges in total.
  - (i) Show that, when the network has  $m$  edges, the probability that the next edge added will attach to vertex  $i$  is  $k_i/m$ .
  - (ii) Write down a master equation giving  $p_k(m+1)$  in terms of  $p_{k-1}(m)$  and  $p_k(m)$ . Give the equation for the special case of  $k = 0$  also.
  - (iii) Eliminate  $m$  from the master equation in favor of the mean degree  $c = 2m/n$  and take the limit  $n \rightarrow \infty$  with  $c$  held constant to show that  $p_k(c)$  satisfies the differential equation

$$c \frac{dp_k}{dc} = (k-1)p_{k-1} - kp_k.$$

- (iv) Define a generating function  $g(c, z) = \sum_{k=0}^{\infty} p_k(c) z^k$  and show that it satisfies the partial differential equation

$$c \frac{\partial g}{\partial c} + z(1 - z) \frac{\partial g}{\partial z} = 0.$$

- (v) Show that  $g(c, z) = f(c - c/z)$  is a solution of this differential equation, where  $f(x)$  is any differentiable function of  $x$ .
- (vi) The particular choice of  $f$  depends on the initial conditions on the network. Suppose the network starts off in a state where every vertex has degree one, which means  $c = 1$  and  $g(1, z) = z$ . Find the function  $f$  that corresponds to this initial condition and hence find  $g(c, z)$  for all values of  $c$  and  $z$ .
- (vii) Show that, for this solution, the degree distribution as a function of  $c$  takes the form

$$p_k(c) = \frac{(c - 1)^{k-1}}{c^k},$$

except for  $k = 0$ , for which  $p_0(c) = 0$  for all  $c$ .

Note that the degree distributions in both this model and the model of question 3 decay exponentially in  $k$ , implying that neither preferential attachment nor network growth alone can account for a power-law degree distribution. One must have both growth and preferential attachment to get a power law.