

Complex Systems 535/Physics 508: Homework 8

1. Consider a configuration model network that has vertices of degree 1, 2, and 3 only, in fractions p_1 , p_2 , and p_3 , respectively.
 - (i) Find the value of the critical vertex occupation probability ϕ_c at which site percolation takes place on the network.
 - (ii) Show that there is no giant cluster for any value of the occupation probability ϕ if $p_1 > 3p_3$. In terms of the structure of the network, why is this? And why does the result not depend on p_2 ?
 - (iii) Find the size of the giant cluster as a function of ϕ . (Hint: you may find it useful to remember that $u = 1$ is always a solution of the equation $u = 1 - \phi + \phi g_1(u)$.)

2. Consider the spread of an SIR-type disease on a network in which some fraction of the individuals have been vaccinated against the disease. We can model this situation using a joint site/bond percolation model in which a fraction ϕ_s of the vertices are occupied to represent the vertices not vaccinated, and a fraction ϕ_b of the edges are occupied to represent the edges along which contact sufficient for disease transmission takes place.
 - (i) Show that the fraction S of individuals infected in the limit of long time is given by the solution of the equations

$$S = \phi_s[1 - g_0(u)], \quad u = 1 - \phi_s\phi_b + \phi_s\phi_b g_1(u),$$
 where $g_0(z)$ and $g_1(z)$ are the generating functions for the degree distribution and excess degree distribution, as usual.
 - (ii) Show that for a given probability of transmission ϕ_b the fraction of individuals that need to be vaccinated to prevent spread of the disease is $1 - 1/[\phi_b g_1'(1)]$.

3. Recall the “acquaintance immunization” process we discussed in class: instead of vaccinating random people, you choose random people and get them to nominate a friend, then you vaccinate the friend. Because your friends tend to be the popular people, this has the beneficial effect of vaccinating people with many contacts.

Consider the acquaintance immunization process on the configuration model.

- (i) With the configuration model, when you follow an edge to a friend, you arrive at vertex i with probability $k_i/2m$. Thus the probability that a person with degree k gets vaccinated in the acquaintance immunization scheme is proportional to k , or, in the notation used in the book, $1 - \phi_k = Ak$ for some constant A , where ϕ_k is the probability that a vertex of degree k is “occupied,” i.e., not vaccinated. If the total fraction of unvaccinated individuals is $\bar{\phi}$ (which is equal to $\sum_k p_k \phi_k$), then find the value of the constant A in terms of $\bar{\phi}$ and hence show that

$$\phi_k = 1 - (1 - \bar{\phi}) \frac{k}{\langle k \rangle}.$$

- (ii) Show that the generating function $f_0(z)$ defined in Eq. (16.32) is given by $f_0(z) = g_0(z) - (1 - \bar{\phi})z g_1(z)$, where $g_0(z)$ and $g_1(z)$ are the generating functions for the degree distribution and excess degree distribution, as usual. Hence find an expression for the function $f_1(z)$ from Eq. (16.36).

- (iii) Using Eq. (16.41), show that in order to eradicate a disease on a configuration model network using acquaintance immunization, the fraction $1 - \bar{\phi}$ of vaccinated individuals must satisfy

$$1 - \bar{\phi} > \frac{\langle k^2 \rangle \langle k \rangle - 2 \langle k \rangle^2}{\langle k^3 \rangle - \langle k^2 \rangle}.$$

This is one of the very few examples I know of in which the third moment of the degree distribution enters a calculation.

4. **Extra credit:** Create a computer simulation, in the computer language of your choice, of the spread of an SIR-type disease over a network. Your program should perform the following steps:

- (i) Generate a Poisson random graph of the $G(n, m)$ type with $n = 10\,000$ vertices $m = 25\,000$ edges, and select one vertex at random to be the single initial carrier of the disease.
- (ii) On each time step, every currently infected vertex spreads the disease to each of its currently susceptible neighbors with independent probability $\phi = 0.4$, then recovers and remains in the recovered state indefinitely thereafter. (This particular variant of SIR, in which vertices remain infective for exactly one time step, is called the Reed–Frost model.)

Run your program for many time steps until no infected vertices remain in the network and make a graph showing, on the same axes, the number of susceptible, infected, and recovered vertices as a function of time. You may want to run the program several times to find a good example to print out. (Sometimes you will find that the disease dies out after only infecting a few individuals and no epidemic occurs.)

For full credit, turn in a printout of your program and a copy of the figure you produced.